CONSTRUCTIONS OF MANY COMPLICATED UNCOUNTABLE STRUCTURES AND BOOLEAN ALGEBRAS

BY
SAHARON SHELAH[†]

ABSTRACT
This article has three aims: (1) To make the results of [12, VIII] on constructing models more available for application, by separating the combinatorial parts. Thus in applications one will only need the relevant things from the area of application. (2) To strengthen the results there. In particular, we were mainly interested in [12, VIII] in showing that there are many isomorphism types of models of an unsuperstable theory, with results about the number of models not elementarily embeddable in each other being a side benefit. Here we consider the latter case in more detail, getting more cases. We also consider some more complicated constructions along the same lines ($K^{\omega}_{\mathrm{ptr}}$). (3) To solve various problems from the list of van Dowen, Monk and Rubin [3] on Boolean algebras, which was presented at a conference on Boolean algebra in Oberwolfach January 1979 (most of the solutions are mentioned in the final version). Some of them are not related to (1) and (2). This continues [10, §2] in which the existence of a rigid B.A. in every uncountable power was proved. There (and also here) we want to demonstrate the usefulness of the methods developed in [12, VIII] (and §§1,2) for getting many (rigid) non-embeddable models in specific classes.

§0. Introduction

In §1 we present the abstract context. Clearly, e.g., K^{ω}_{tr}, ψ_{tr}, $K^{\omega}_{\mathrm{ptr}}$, ψ_{ptr} are variants of the same idea. It is, however, not clear whether it is worthwhile now to find a common generalization. This section contains mainly definitions.

In §2 we do the combinatorial part; the point is that, in order to apply the method, no understanding of the proof of these combinatorial facts is required (though if you need another pair K, ψ you may well have to understand them in

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order to do the required changes). This section can therefore be waived by a reader interested in applications only.

At present, some of the constructions from [12] are not represented, in particular on K_{or} (on which the results are naturally stronger), and also K^{κ}_{tr}, $\kappa > \aleph_0$.

We prove, e.g. (by 2.1, 2.2, 2.6, 2.7, 2.8),

0.1. THEOREM. *If λ is regular, $\lambda > \mu \geqq \aleph_0$ or if $\lambda^{\aleph_0} = \lambda$, and $\mu^{\aleph_0} < \lambda$, then K has the full strong $(\lambda, \lambda, \mu, \aleph_0)$-$\psi$-bigness property for $(K, \psi) = (K^{\omega}_{\mathrm{tr}}, \psi_{\mathrm{tr}})$, and, if in addition for regular λ, $\wedge(\forall \theta < \lambda)\ [\theta^{\aleph_0} < \lambda]$, also for $(K, \psi) = (K^{\omega}_{\mathrm{ptr}}, \psi_{\mathrm{ptr}})$. If $\mu \leqq \chi < \lambda$, $\chi^{\aleph_0} = \chi$, $2^{\chi} \geqq \lambda$ the full strong $(\lambda, \lambda, \mu, \aleph_0)$-$\psi_x$-bigness property holds for $x = \mathrm{tr}, \mathrm{ptr}$.*

We can replace $(\lambda, \lambda, \mu, \aleph_0)$ by $(\lambda, \lambda, \mu, \kappa)$, for functions which are strongly finitary on P_ω, if $\mu^{<\kappa} < \lambda$, and in the last case $\mu^{<\kappa} \leqq \chi$. Note that without κ the case left out is: $\beth_\delta \leqq \lambda < \beth_{\delta+1}$, δ is zero or of cofinality $\aleph_0$; and λ is singular (for $K^{\omega}_{\mathrm{tr}}, \psi_{\mathrm{tr}}$)).

In §3 we apply §1, §2 to Boolean algebras.

0.2. CONCLUSION. If $\lambda^{\aleph_0} = \lambda > 2^{\aleph_0}$ then there is a rigid complete Boolean algebra of power λ which satisfies the countable chain condition, and is mono-rigid (= no non-trivial one-to-one endomorphism).

PROOF. See 0.1, 3.16 and 6.3 for $\lambda > 2^{\aleph_0}$.

0.3. CONCLUSION. For $\lambda > \aleph_0$ there is a mono-rigid B.A. of power λ satisfying the countable chain condition.

PROOF. See 0.1, 3.13, 6.11, 6.12(1).

0.4. CONCLUSION. If $\lambda > \aleph_0$ then there is a Bonnet-rigid (hence onto rigid and mono-rigid) B.A. of power λ.

PROOF. See 0.1, 6.4, 6.10, 6.12(2).

In §4, we prove two results on Boolean algebras unrelated to §1, §2, §3: let us call B λ-narrow if among any λ elements of B, two are comparable. We prove that any Boolean algebra which has no dense subset of power $< \lambda$, is not λ-narrow (extending a result of Baumgartner and Komjath [1] for λ regular).

The second result is: for λ singular of cofinality $> \aleph_0$, there *may* be a Boolean algebra which is λ-narrow but not μ-narrow for $\mu < \lambda$.

Lastly, in §5 assuming $\diamondsuit_{\aleph_1}$, we construct, by [3] notation, a Bonnet-rigid, endo-rigid indecomposable B.A. of power $\aleph_1$. For this we introduce a new notion: "a Boolean algebra B *absolutely omit* a type p" (see [18] for a more general discussion).

In §6, we close some loose ends: We deal with a variant $\mathrm{tr}(\alpha)$ of ptr, and deal with a variant of bigness (ind_χ-big).

By the proofs in [12, VIII §2] and 0.1 it is clear that

0.5. CONCLUSION. (1) If K^{ω}_{tr} has the strong $(2^\lambda, \lambda, \mu, \aleph_0)$-$\psi_{\mathrm{tr}}$-bigness property $T \subseteq T_1$, T unsuperstable, $|T_1| \leqq \mu$ then $\mathrm{IE}(\lambda, T_1, T) = 2^\lambda$.

(2) This holds when $\lambda > \mu$, except possibly when: $\beth_\delta \leqq \lambda < \beth_{\delta+1}$, δ limit, $\mathrm{cf}\,\delta = \aleph_0$ or δ is zero, λ singular, and $(\forall \kappa < \lambda) 2^\kappa < 2^\lambda$.

NOTATION. We let m, n, k, l be natural numbers, $i, j, \alpha, \beta, \gamma, \delta, \xi, \zeta$ be ordinals, where δ is reserved for limit ordinals. Let $\lambda, \mu, \kappa, \chi$ be cardinals, usually infinite. Let η, ν, ρ be sequences of ordinals, $l(\eta)$ the length of η, $\eta(i)$ the ordinal in the i-th place of η, $\eta\,\hat{}\,\nu$ the concatenation.

A bar on a letter means we have a sequence of elements of this kind. If x_n is defined, $\bar{x}_{\bar{n}} = \bar{x}[\bar{n}] = (x_{n(0)}, x_{n(1)}, \cdots,)$; $\bar{x} = \langle x_0\, x_1, \cdots, x_{n-1} \rangle$.

Let ${}^\beta\alpha$ be the set of sequences of length β of ordinals $< \alpha$. Let ${}^{\beta>}\alpha = \bigcup_{\gamma<\beta} {}^\gamma\alpha$, ${}^{\beta\geqq}\alpha = \bigcup_{\gamma\leqq\beta} {}^\gamma\alpha$.

DEFINITION. For η a sequence of ordinals let $\mathrm{orco}(\eta)$ (the ordinal content of η) be $\{\eta(i) : i < l(\eta)\}$. Also for η a sequence of ordinals and sequences of ordinals let

$$\mathrm{orco}(\eta) = \{\alpha : \text{ for some } i,\ \eta(i) = \alpha \text{ or for some } i,\ l < n,\ \eta(i) = \langle \alpha_0, \cdots, \alpha_{n-1} \rangle,\ \alpha = \alpha_l\}.$$

Similarly $\mathrm{orco}(\bar{\eta}) = \bigcup_{i<l(\bar{\eta})} \mathrm{orco}(\eta_i)$.

BOOLEAN ALGEBRAS. We abbreviate Boolean Algebra by B.A., and use the letter B for such an algebra. A Boolean Algebra has an individual constant 0, and the operation $a \cup b$, $a \cap b$ and $a - b$. A function f from B_1 to B_2 is an embedding if it is one to one and preserves those operations. So a B.A. B has a maximal element 1_B, but it is not necessarily preserved by an embedding. The infinitary operations are defined similarly, and f is a σ-embedding if it commutes with $\bigcap_{n<\omega}$, $\bigcup_{n<\omega}$ too.

For a set A of elements of a B.A., $\langle A \rangle_B$ is the subalgebra generated by $A \cup \{1_B\}$, and $B \restriction b = \{a : a \leqq b\}$ ($a \leqq b$ means $a \cup b = b$), so $1_{B \restriction b} = b$.

Let $-b$ be $1 - b$; and let b^ε be b if $\varepsilon = 0$, $-b$ if $\varepsilon = 1$.

§1. The framework

1.1. CONTEXT. Let κ, μ be infinite cardinals, κ regular, $F_{\alpha,\beta}$ a β-place function symbol, for $\alpha < \mu$, $\beta < \kappa$. $L = \{F_{\alpha,\beta} : \alpha < \mu, \beta < \kappa\}$, so $L = L(\mu, \kappa)$.

Let K be a class of models which we shall call index models (i.e., they serve as a set of indices). Members of K shall be denoted by I, J perhaps with indices. For $I_\alpha \in K$, $\alpha < \alpha(0)$, $\Sigma_\alpha I_\alpha$ is the model in K whose set of elements is $\bigcup_\alpha I_\alpha \times \{\alpha\}$, the relation R is $\{\langle t_1, \alpha\rangle \cdots \langle t_n, \alpha\rangle : I_\alpha \models R[t_1 \cdots]\}$; similarly for functions (so we allow partial functions) and individual constants are treated as one-place functions, and we have the relation $E = \{\langle\langle t, \alpha\rangle, \langle s, \alpha\rangle\rangle : s, t \in I_\alpha\}$.

For $I \in K$, $M(I)$ is the free L-algebra generated by I, expanded by the relations and functions of I, and P, where $P^{M(I)} = I$. $M^*(I)$ is any expansion of $M(I)$ by a well ordering $<^*$ of $|M(I)| - |P|$, such that $\tau = F_{\alpha,\beta}(\tau_1, \cdots)$ implies $\tau_i <^* \tau$. So $M^*_{\mu,\kappa}(I)$ is not uniquely defined.

The support of $\tau(a_0, \cdots) \in M(I)$ is $\{a_0, \cdots\} \subseteq I$.

If the identity of μ, κ may not be clear we shall write $M_{\mu,\kappa}(I)$, $M^*_{\mu,\kappa}(I)$.

Let $\bar{a}, \bar{b}$ denote sequences of elements, usually of I or $M(I)$. A term of $L(\mu, \kappa)$ is denoted by $\tau(\bar{x})$, $\bar{x}$ a sequence of variables of length $< \kappa$. Let $\bar{\tau}(\bar{x}) = (\tau_0(\bar{x}), \tau_1(\bar{x}), \cdots)$. Hence $\bar{a} = \bar{\tau}(\bar{b})$ means $a_i = \tau_i(\bar{b})$. Note that every sequence $\bar{a}$ from $M(I)$ of length $< \kappa$ has a representation $\bar{\tau}(\bar{b})$, $\bar{b} \in {}^{\kappa>}|I|$.

We call $\bar{a}$ *finitary* if $\bar{a} = \bar{\tau}(\bar{b})$, $\bar{b} \in {}^{\omega>}|I|$: we call it strongly finitary if in addition τ has finitely many subterms.

1.2. DEFINITION. We say that $I \in K$ is $\psi(\bar{x}, \bar{y})$-unembeddable in $J \in K$ provided that: if f is a function from I to $M(J)$, $A \subseteq J$, $|A| < \kappa$ *then* for some sequences $\bar{a}, \bar{b}$ from I (i.e. of elements of I), of length $l(\bar{x}) = l(\bar{y})$, $I \models \psi[\bar{a}, \bar{b}]$ and $f(\bar{a}) = \bar{\tau}(c_0, \cdots)$, $f(\bar{b}) = \bar{\tau}(d_0, \cdots)$, $c_i, d_i \in J$ and $\langle c_0, \cdots\rangle$, $\langle d_0, \cdots\rangle$ realize the same quantifier free type over A in J. For such $f(\bar{a})$, $f(\bar{b})$ we write $f(\bar{a}) \approx_A f(\bar{b}) \bmod M(J)$. If the identity of μ, κ is not clear we write (μ, κ, ψ) instead of ψ. If we write $\psi(\bar{x}, \bar{y}, \bar{z}, \bar{v}, \cdots)$, the meaning is clear; similarly the meaning of ψ-embeddable. We omit ψ if it is $\wedge\{\varphi(\bar{x}) \equiv \varphi(\bar{y}) : \varphi$ quantifier free$\}$.

1.3. DEFINITION. We say that $I \in K$ is strongly $\psi(\bar{x}, \bar{y})$-unembeddable in $J \in K$ provided that for any $M^*(J)$, if f is a function from I to $M^*(J)$, $A \subseteq J$, $|A| < \kappa$ *then* for some sequences $\bar{a}, \bar{b}$ from I, $\models \psi[\bar{a}, \bar{b}]$, $f(\bar{a}) = \bar{\tau}(c_0, \cdots)$, $f(\bar{b}) = \bar{\tau}(d_0, \cdots)$, $\langle c_0, \cdots\rangle$, $\langle d_0, \cdots\rangle$ realize the same quantifier free type over A in J, and for any subterms τ_1, τ_2 of $\bar{\tau}$, and $\sigma(a_0, \cdots) \in M(J)$, $a_0, \cdots \in A$, $\tau_1(c_0, \cdots) <^* \sigma(a_0, \cdots)$ iff $\tau_1(d_0, \cdots) <^* \sigma(a_0, \cdots)$; similarly for $*>$, and $\tau_1(c_0, \cdots) <^* \tau_2(c_0, \cdots)$ iff $\tau_1(d_0, \cdots) <^* \tau_2(d_0, \cdots)$. For such $f(\bar{a}), f(\bar{b})$ we write $f(\bar{a}) \approx_A f(\bar{b}) \bmod M^*(I)$. We use the abbreviation of 1.2.

REMARKS. (1) So strongly embeddable is a weaker notion than embeddable.

(2) The "over A" in Definitions 1.2 and 1.3 can be omitted without harming the paper.

1.4. DEFINITION. (1) K has the $(\chi, \lambda, \mu, \kappa)$-$\psi$-bigness property if there are $I_i \in K$ $(i < \chi)$, $|I_i| = \lambda$, $L = L(\mu, \kappa)$ (see 1.1) and for $i \neq j$, I_i is ψ-unembeddable into I_j.

(2) K has the strong $(\chi, \lambda, \mu, \kappa)$-$\psi$-bigness property if there are $I_i \in K$ $(i < \chi)$, $|I_i| = \lambda$, $L = L(\mu, \kappa)$ (see 1.1) and for $i \neq j$, I_i is strongly ψ-unembeddable into I_j.

(3) We add in the notions above "the full ..." if we replace I_j by $\Sigma_{\alpha \neq i, \alpha < \chi} I_\alpha$.

(4) We add in the notion "... property for f such that ..." if we modify accordingly the unembeddability, i.e., restrict ourselves to functions f satisfying....

(5) We also say that the pair (K, ψ) has such properties.

(6) We say that K is almost closed under sums for λ (and ψ) if *for every* $I_\alpha \in K$ (for $\alpha < \alpha_0 \leqq \lambda$), I_α of power $\leqq \lambda$, *there are* J, g, h_α $(\alpha < \alpha_0)$ such that:

(a) $J \in K$, $|J| = \lambda$,

(b) $h_\alpha : I_\alpha \to J$ and for any $x_0, \cdots, y_0, \cdots \in I_\alpha$,
$I_\alpha \vDash \psi[\langle x_0, \cdots \rangle, \langle y_0, \cdots \rangle]$ implies $J \vDash \psi[\langle h_\alpha(x_0), \cdots \rangle], \langle h_\alpha(y_0), \cdots \rangle]$,

(c) $g : J \to \Sigma_{\alpha < \alpha_0} I_\alpha$ such that, defining

$$R = \{\langle\langle \eta, i\rangle, \langle \nu, j\rangle\rangle : \eta \in I_i, \nu \in I_j \text{ and } i < j\} \subseteq \Big(\sum_{\alpha<\alpha_0} I_\alpha\Big) \times \Big(\sum_{\alpha<\alpha_0} I_\alpha\Big),$$

the following holds:

for any $x_0, \cdots, y_0, \cdots \in J$, *if* $\langle x_0, \cdots \rangle, \langle y_0, \cdots \rangle$ realize the same quantifier free type in J, *then* $\langle g(x_0), \cdots \rangle, \langle g(y_0), \cdots \rangle$ realize the same quantifier free type in $(\Sigma_{\alpha<\alpha_0} I_\alpha, R)$.

We say K is essentially closed under sums for λ if, in addition, Rang h_α, Rang g is a union of equivalence classes of $\approx \bmod J$, $\approx \bmod (\Sigma_{\alpha<\alpha_0} I_\alpha, R)$ resp.

REMARK. We could have made, e.g., $h_\alpha : I_\alpha \to M^*(J)$, *or* in the definition of sum expansion by R, without serious changes in the paper.

1.5. CLAIM. (1) *If K is closed under sums, then the full (strong) $(\chi, \lambda, \mu, \kappa)$-$\psi$-bigness property implies the (strong) $(\chi_1, \lambda, \mu, \kappa)$-$\psi$-bigness property, where* $\chi_1 = \min\{2^\chi, 2^\lambda\}$.

(2) *In* (1), *the "strong" version instead of "K closed under sums" it is enough to assume that K is essentially closed under sums for* λ, ψ.

(3) *The classes defined below* $(K^\kappa_{tr}, K_{or}, K^\kappa_{ptr})$ *are almost closed under sums.*

(4) *The relations above* (*in* 1.4) *have obvious monotonicity properties in* χ, μ, κ; *and for all our* K, *for* λ *too. For example*

$$\chi \leqq \chi' \Rightarrow [(\chi', \lambda, \mu, \kappa)\text{-}bigness \Rightarrow (\chi, \lambda, \mu, \kappa)\text{-}bigness],$$
$$\mu \leqq \mu', \kappa \leqq \kappa' \Rightarrow [(\chi, \lambda, \mu', \kappa')\text{-}bigness \Rightarrow (\chi, \lambda, \mu, \kappa)\text{-}bigness].$$

PROOF. (1) Assume K has the full $(\chi, \lambda, \mu, \kappa)$-$\psi$-bigness property.

Case 1. $\chi \leqq \lambda$. For $\Gamma \subseteq \chi$ let

$$J_\Gamma = \sum_{\alpha \in \Gamma} I_\alpha.$$

Let H be a collection of subsets of χ such that $|H| = 2^\chi$ and $\Gamma \neq \Delta \in H \Rightarrow \Gamma \nsubseteq \Delta$. Suppose Γ, $\Delta \in H$, $f: J_\Gamma \to M(J_\Delta)$. Choose $\alpha \in \Gamma - \Delta$. Thus $f \restriction I_\alpha : I_\alpha \to M(\Sigma_{\beta \neq \alpha} I_\beta)$ and the desired conclusion follows.

Case 2. $\lambda < \chi$. Take H a family of subsets of λ and proceed as in Case 1.

(2) As K has the full $(\chi, \lambda, \mu, \kappa)$-$\psi$-bigness property there are I_α $(\alpha < \chi)$ in K, each of power λ, such that I_α is ψ-unembeddable into $\Sigma_{\beta \neq \alpha} I_\beta$. By the assumption of (2) (K is essentially closed under sums) for every $\Gamma \subseteq \chi$, $|\Gamma| \leqq \lambda$ let J_Γ, g^Γ, h^Γ_α $(\alpha \in \Gamma)$ satisfy (a), (b), (c) of Definition 1.4(6) for $\Sigma_{\alpha \in \Gamma} I_\alpha$. As in the proof of (1) it suffices to show: (*) if Γ, $\Delta \subseteq \chi$, $|\Gamma| \leqq \lambda$, $|\Delta| \leqq \lambda$, $\Gamma - \Delta \neq \emptyset$, $f: J_\Gamma \to M^*(J_\Delta)$, $A \subseteq J_\Delta$, $|A| < \kappa$, *then* for some $\bar{a}, \bar{b} \in J_\Gamma$, $J_\Gamma \vDash \psi[\bar{a}, \bar{b}]$ and $f(\bar{a}) \approx_A f(\bar{b}) \bmod M(J_\Delta)$.

Choose $\alpha \in \Gamma - \Delta$. Let $F^* \in L(\mu, \kappa)$ be a one-place function symbol. We can define a model $m^{*\prime}(\Sigma_{i \in \Delta} I_i)$ so that:

for every $x, y \in I_\alpha$, $\langle x, y \rangle \in R$ implies $M^*(\Sigma_{i \in \Delta} I_i) \vDash F^*(x) <^* F^*(y)$,

for every $x_0, \cdots, y_0, \cdots \in I_\alpha$ and terms τ, σ $M^{*\prime}(\Sigma_{i \in \Delta} I_i \vDash \tau(x_0, \cdots) <^* \sigma(y_0, \cdots)$ implies $M^{*\prime}(\Sigma_{i \in \Delta} I_i) \vDash \tau(F^*(x_0), \cdots) <^* \sigma(F^*(y_0), \cdots)$. Now define $g_\Delta : M^*(\Sigma_{i \in \Delta} J_i) \to M^{*\prime}(\Sigma_{i \in \Delta} I_i)$ by $g_\Delta(\tau(x_0, \cdots)) = \tau(F^*(x_0), \cdots)$. Let $g^*_\Delta = g_\Delta g^\Delta$.

Consider the sequence of mappings:

$$I_\alpha \xrightarrow[h^\Gamma_\alpha]{} J_\Gamma \xrightarrow[f]{} M^*(J_\Delta) \xrightarrow[g^*_\Delta]{} M^{*\prime}\left(\sum_{i \in \Delta} I_i\right).$$

So $g^*_\Delta f h^\Gamma_\alpha : I_\alpha \to M^{*\prime}(\Sigma_{i \in \Delta} I_i)$. As $\Sigma_{i \in \Delta} I_i$ is a submodel of $\Sigma_{i \neq \alpha} I_i$, also w.l.o.g. $M^*(\Sigma_{i \in \Delta} I_i)$ is a submodel of $M^*(\Sigma_{i \neq \alpha} I_i)$; but we know I_α is ψ-unembeddable into $\Sigma_{i \neq \alpha} I_i$. Hence there are $\bar{x}$, $\bar{y} \in I_\alpha$ such that

(i) $I_\alpha \vDash \psi[\bar{x}, \bar{y}]$,

(ii) $g^*_\Delta f h^\Gamma_\alpha(\bar{x}) \approx_{g^*_\Delta(A)} g^*_\Delta f h^\Gamma_\alpha(\bar{y}) \bmod M^{*\prime}(\Sigma_{i \in \Delta} I_i)$.

By (i) and (b) from 1.4 (6),

(iii) $J_\Gamma \models \psi[\bar{x}', \bar{y}']$ where $\bar{x}' = h^\Gamma_\alpha(\bar{x})$, $\bar{y}' = h^\Gamma_\alpha(\bar{y})$.

By (ii) and the definition of $\bar{x}'$, $\bar{y}'$

(iv) $g^*_\Delta(f(\bar{x}')) \approx_{g^*_\Delta(A)} g^*_\Delta(f(\bar{y}')) \bmod M^{*\prime}(\Sigma_{i\in\Delta} I_i)$.

By (iv), (c) of 1.4 (6), the definition of $M^*(\Sigma_{i\in\Delta} I_i)$, and of g^*_Δ, g_Δ,

(v) $f(\bar{x}') \approx_A f(\bar{y}') \bmod M^*(J_\Delta)$.

So we have proved (*) (by (iii) and (v)) which suffices.

(3)–(4) Left to the reader.

Now we define the K and ψ in which we shall be interested:

1.6. DEFINITION. (1) K^κ_{tr} is the class of I, such that:

(a) The universe of I is a subset of ${}^{\kappa\geqq}\lambda$ for some λ, closed under initial segments.

(b) The relations of I are $P_i = \{\eta \in I : l(\eta) = i\}$, for all $i \leqq \kappa$, $\lessdot$ where $\eta \lessdot \nu$ iff $\eta = \nu \restriction l(\eta)$ and $< = \{\langle \eta^\frown\langle\alpha\rangle, \eta^\frown\langle\beta\rangle\rangle : \eta^\frown\langle\alpha\rangle \in I, \eta^\frown\langle\beta\rangle \in I, \alpha < \beta\}$ and $\mathrm{Eq}_i = \{\langle \eta, \nu\rangle : \eta \restriction i = \nu \restriction i, \eta \in I, \nu \in I\}$ and an individual constant $\langle\ \rangle$ denoting the empty sequence.

(2) We let $\psi_{tr}(\bar{x}, \bar{y}) = \bigvee_{i+1<\kappa} [P_{i+1}(x_0) \wedge P_{i+1}(y_0) \wedge P_\kappa(x_1) \wedge \mathrm{Eq}_i(x_0, y_0) \wedge x_0 \neq y_0 \wedge x_1 = y_1 \wedge x_0 \lessdot x_1 \wedge y_0 < x_0]$.

(3) We let $\psi_{trnh}(\bar{x}, \bar{y}) = [\bigwedge_n (x_n \,\mathrm{Eq}_n x_{n+1} \wedge P_n(x_n) \wedge y_n \,\mathrm{Eq}_n y_{n+1} \wedge P_n(y_n)) \wedge (\exists x) \bigwedge_n x \,\mathrm{Eq}_n x_n \wedge \neg (\exists y) \bigwedge_n y \,\mathrm{Eq}_n y_n]$.

REMARKS. (1) We can replace ψ_{tr} by

$$\bigvee_{i+1<\kappa} \Big[\bigwedge_{j\leqq i+1} (P_j(x_j) \wedge P_j(y_j)) \wedge \bigwedge_{j\leqq i} x_j = y_j \wedge x_{i+r} = y_{i+r} \wedge P_\kappa(x_{i+r}) \wedge \bigwedge_{\alpha<\beta\leqq\alpha+1} (x_\alpha \lessdot x_\beta \lessdot x_{i+2} \wedge y_\alpha \lessdot y_\beta) \wedge x_{i+1} \neq y_{i+1} \Big]$$

without changing much the paper.

A similar remark holds for ψ_{ptr} (see Definition 1.8 below).

(2) We thought that the existence of the full strong $(\lambda, \lambda, \aleph_0, \aleph_0)$-$\psi_{tr}$-bigness property of κ^ω_{tr} extracts the combinatorial content of [12] and the related constructions, but a question of Grossberg reveals that it seems that is not the case, concerning [20], theorem 1.2. Consideration of this suggests:

DEFINITIONS. (1) $I \in \kappa^\omega_{tr}$ is *-unembeddable into $J \in \kappa^\omega_{tr}$, *if* when χ is a regular cardinal, $<$ a well ordering of $H(\chi)$, I, J, f, belong to $H(\chi)$, $f : I \to H(\chi)$, and $p \in H(\chi)$ and we define, for $\eta \in {}^{\omega>}\lambda$, $N_\eta < (H(\chi), \in, <)$ by induction on $l(\eta)$ as the Skolem hull in $(H(\chi), \in, <)$ of $\{p\} \cup \{\eta(l) : l < l(\eta)\} \cup \{N_{\eta\restriction l} : l < l(\eta)\}$, *then* for some $\eta \in I$, $l(\eta) = \omega$ and for every $\nu \in J$ of length ω either $(\exists k < \omega)[\nu \restriction k \notin N_\eta]$ or $(\exists l < \omega)(\forall k < \omega)[\nu \restriction h \in N_{\eta\restriction l}]$.

(2) We define $(\mu,\kappa,*)$-unembeddability similarly, only N_η is the smallest elementary submodel of $(H(\chi),\in,<)$ including $\{i : i<\mu\}\cup\{p\}\cup\{\eta(l), N_{\eta\restriction l} : l<l(\eta)\}$ and closed under taking sequences of length $<\kappa$.

(3) The (full) (strong) $(\lambda,\chi,\mu,\kappa)$-*-bigness property is defined as in Definition 1.4. Now the proofs in §2 on $(\kappa^{\omega}_{\mathrm{tr}},\psi_{\mathrm{tr}})$ work also for this notion: it implies the previous one; and we can make similar changes to bigness properties of $(\kappa^{\omega}_{\mathrm{ptr}},\psi_{\mathrm{ptr}})$.

1.7. Definition. Let K_{or} be the class of linear orders $\psi_{\mathrm{or}}(\bar x,\bar y)=[x_0<x_1\equiv y_1<y_0]\wedge x_0\neq x_1\wedge y_0\neq y_1$.

1.8. Definition. $K^{\kappa}_{\mathrm{ptr}}$ is the class of I such that:

(a) The set of elements of I is a subset of $\{\eta : \eta$ is a sequence of length $\leqq\kappa$, for $i+1<l(\eta)$, $\eta(i)=\langle\alpha,\beta\rangle$, $\alpha<\beta$, and for $i+1=l(\eta)$, $\eta(i)$ is an ordinal$\}$. Also if $\eta\in I$, $i+1<l(\eta)$, $\eta(i)=\langle\alpha,\beta\rangle$ then $(\eta\restriction i)^\wedge\langle\alpha\rangle\in I$ and $(\eta\restriction i)^\wedge\langle\beta\rangle\in I$.

(b) The relations of I are: $\eta\ll\nu\overset{\mathrm{def}}{=}\eta=\nu\restriction l(\eta)$, $P_i=\{\eta : l(\eta)=i\}$,

$<^*_i=\{\langle\eta,\nu\rangle : l(\eta)=l(\nu)=i+1, \eta(i)<\nu(i), \eta\restriction i=\nu\restriction i\}$,
$\mathrm{Eq}_i=\{\langle\eta,\nu\rangle : \eta\restriction i=\nu\restriction i\}$

and $\mathrm{Suc}_L=\{\langle\eta,\nu\rangle : \eta\restriction i=\nu\restriction i, i+1=l(\eta), \nu(i)=\langle\alpha,\beta\rangle, \eta(i)=\alpha\}$,
$\mathrm{Suc}_R=\{\langle\eta,\nu\rangle : \eta\restriction i=\nu\restriction i, i+1=l(\eta), \nu(i)=\langle\alpha,\beta\rangle, \eta(i)=\beta\}$,
an individual constant $\langle\ \rangle$, and functions
$\mathrm{Res}^L_n(\eta)=\eta\restriction n^\wedge\langle\alpha\rangle\ \mathrm{Res}^R_n(\eta)=\eta\restriction n^\wedge\langle\beta\rangle$ if $\eta(n)=\langle\alpha,\beta\rangle$,
$\psi_{\mathrm{ptr}}(\bar x,\bar y)=\bigvee_{i+1<\kappa}[P_{i+1}(x_0)\wedge P_{i+1}(y_0)\wedge P_\kappa(x_1)\wedge x_1=y_1\wedge\mathrm{Suc}_L(x_0,x_1)\wedge\mathrm{Suc}_R(y_0,y_1)\wedge x_0<^*_i y_0]$.

1.9. Definition. For $I\in K^{\kappa}_{\mathrm{tr}}$, $\bar a$, $\bar b\in{}^{\kappa>}|M_{\mu,\kappa}(I)|$, α an ordinal, $\bar a\approx\bar b \bmod (M_{\mu,\kappa}(I),\alpha)$ if $\bar a=\bar\tau(c_0,\cdots)$, $\bar b=\bar\tau(c_0,\cdots)$. $\bar a\approx\bar b\bmod M_{\mu,\kappa}(I)$, and for any i, and $\xi<l(c_i)$ if $c_i(\xi)<\alpha\vee d_i(\xi)<\alpha$ then $c_i(\xi)=d_i(\xi)$. Similarly for $M^*_{\mu,\kappa}(I)$ and for $K^{\kappa}_{\mathrm{ptr}}$.

1.10. Definition. (1) A model N is representable in $M(I)$ if there is a function $f:|N|\to M(I)$ such that if $a_0,\cdots,a_{n-1}$, $b_0,\cdots,b_{n-1}\in N$ and $\langle f(a_0)\cdots\rangle\approx\langle f(b_0),\cdots\rangle \bmod M(I)$ then $\langle a_0,\cdots\rangle$, $\langle b_0,\cdots\rangle$ realize the same quantifier free type in N.

(2) We can replace everywhere above $M(I)$ by $M^*(I)$.

(3) We say "N is strictly representable in $M(I)$" if in addition $a_0\approx b_0\bmod M(I)$ implies $a_0\in\mathrm{Range}(f)\Leftrightarrow b_0\in\mathrm{Range}(f)$.

§2. Constructions for proving the bigness property

We try here to get theorems in ZFC: extra set theoretic assumptions of course simplify the situation (e.g., Theorem 2.1 holds for λ singular if $\{\mu < \lambda : \mu$ is regular and there is stationary $S \subseteq \{\delta < \mu : \mathrm{cf}\,\delta = x_0\}$ with no initial segment stationary$\}$ is unbounded below λ).

2.1. THEOREM. *If* $\aleph_0 < \lambda$, λ *is regular,* $\mu < \lambda$, $\lambda \leqq \lambda^*$ then K^{ω}_{tr} *has the full strong* $(\lambda, \lambda^*, \mu, \aleph_0)$-$\psi_{\mathrm{tr}}$*-bigness property.*

PROOF. In fact this is proved in [12, VIII 2.1], but similar theorems are proved here later, e.g. 2.2.

2.2. THEOREM. *If* $\mu < \lambda \leqq \lambda^*$, $(\forall \theta < \lambda)[\theta^{\aleph_0} < \lambda]$, λ *regular then* $K^{\omega}_{\mathrm{ptr}}$ *has the full strong* $(\lambda, \lambda^*, \mu, \aleph_0)$-$\psi_{\mathrm{ptr}}$*-bigness property. Moreover, it has the full strong* $(\lambda, \lambda^*, \mu, \kappa)$*-bigness property for f such that* $[P_\omega(a) \Rightarrow f(a)$ *strongly finitary*$]$ *provided that* $\mu^{<\kappa} < \lambda$.

REMARK. We can replace "strongly finitary" by finitary if we are content with "the full $(\lambda, \lambda, \mu, \kappa)$-bigness ..." or we change slightly the definition of $M^*(I)$. Similar remarks apply to the other theorems in this section.

PROOF. It is well known that $S = \{\delta < \lambda : \delta$ a limit ordinal of cofinality $\aleph_0\}$ is stationary. Hence by a well known theorem of Solovay there are stationary pairwise disjoint $S_i \subseteq S$ (for $i < \lambda$), $S = \bigcup_{i<\lambda} S_i$. Let

$$I_i = \{\langle\langle\alpha_0, \beta_0\rangle, \cdots, \langle\alpha_{n-2}, \beta_{n-2}\rangle, \alpha_{n-1}\rangle :$$
$$\alpha_l, \beta_l \text{ ordinals} < \lambda^*, \alpha_l < \beta_l < \alpha_{l+1} \text{ for } l < n-1 \text{ and } n < \omega\}$$
$$\cup \{\langle\langle\alpha_0, \beta_0\rangle, \cdots, \langle\alpha_n, \beta_n\rangle, \cdots\rangle : \alpha_n < \beta_n < \alpha_{n+1} < \lambda, \text{ and } \textstyle\bigcup_{n<\omega}\alpha_n \in S_i\}$$

and $L = L(\mu, \kappa)$. Trivially $|I_i| = \lambda^*$, as $\lambda^{\aleph_0} = \lambda$.

We shall prove that I_i is strongly ψ_{ptr}-unembeddable into $I_i^- = \Sigma_{j \neq i} I_j$, by an f such that $[a \in P^{I_i}_{\omega} \Rightarrow f(a)$ is strongly finitary$]$, assuming $\mu^{<\kappa} < \lambda$, thus finishing the proof. We concentrate on the case $\lambda = \lambda^*$.

So suppose f is a function from I_i into $M^*(I_i^-)$ where $f(\eta)$ is strongly finitary for η of length ω.

For any $a \in I_i$ let $f(a) = \tau_a(\bar{c}_a)$ and $\alpha(a) = \sup\{\gamma < \lambda : \gamma$ appears in $\bar{c}_a\}$; clearly $\alpha(a) < \lambda$. Hence the set $C = \{\delta < \lambda :$ if $a \in I$, $\mathrm{orco}(a) \leqq \delta$ then $\alpha(a) < \delta\}$ is a closed unbounded subset of λ. Also for any $\eta = \langle\langle\alpha_0, \beta_0\rangle, \cdots \langle\alpha_{n-1}, \beta_{n-1}\rangle\rangle$, $\alpha_0 < \beta_0 < \alpha_1 < \cdots < \beta_{n-1} < \lambda$, the following equivalence relation of $\lambda : \alpha E^f_\eta \beta$ iff $\langle \cdots, f(\eta \upharpoonright l^\wedge\langle\alpha_l\rangle),\ f(\eta \upharpoonright l^\wedge\langle\beta_l\rangle) \cdots,\ f(\eta^\wedge\langle\alpha\rangle)\rangle \approx \langle \cdots,\ f(\eta \upharpoonright l^\wedge\langle\alpha_l\rangle),\ f(\eta \upharpoonright l^\wedge\langle\beta_l\rangle),$

$\cdots, f(\eta^\wedge\langle\beta\rangle)\rangle \bmod (M^*(I_i^-), \beta_{n-1})$ has $<\lambda$ equivalence classes. Hence for every $\gamma<\lambda$ there is $\beta(\eta,\gamma)<\lambda$, such that for any β_1, $\gamma<\beta_1<\lambda$ there is β_2, $\gamma<\beta_2<\beta(\eta,\gamma)$ s.t. $\beta_1 E^f_\eta \beta_2$. Again the following is a closed unbounded subset of λ :

$$C_1=\{\delta\in C: \text{if } \alpha_0<\beta_0<\alpha_1<\cdots<\beta_{n-1}<\delta, n<\omega,$$
$$\eta=\langle\langle\alpha_0,\beta_0\rangle,\cdots\langle\alpha_{n-1},\beta_{n-1}\rangle\rangle, \gamma<\delta \text{ then } \beta(\eta,\gamma)<\delta\}.$$

So we can find $\delta\in C_1\cap S_i$ (remember $S_i\subseteq S$ is stationary); so δ has cofinality $\aleph_0$, and let $\delta_n<\delta_{n+1}<\delta$, $\delta=\bigcup_{n<\omega}\delta_n$. Now we define by induction on n, α_n, β_n, such that:

(*) if $\eta_n=\langle\langle\alpha_0,\beta_0\rangle,\cdots,\langle\alpha_{n-1},\beta_{n-1}\rangle\rangle$, then $\alpha_n E^f_{\eta_n}\beta_n$, and $\delta_n, \beta_{n-1}<\alpha_n<\beta_n<\delta$.

This is easily done — we let $\gamma_n=\max\{\delta_n,\beta_{n-1}\}$, $\beta_n=\beta(\eta_n,\gamma_n)+1$, and α_n is chosen by the definition of $\beta(\eta_n,\gamma_n)$ such that $\gamma_n<\alpha_n<\beta(\eta_n,\gamma_n)$, $\beta_n E^f_{\eta_n}\alpha_n$.

Let η be the unique $\eta\in I_i\cap{}^\omega(\lambda\times\lambda)$, $\eta\restriction n=\eta_n$ for every n (η exists by the definition of I_i) and let $f(\eta)=\tau_\eta(\langle\nu_0,j_0\rangle,\cdots,\langle\nu_{k-1},j_{k-1}\rangle)$, where $j_l\neq i$, $j_l<\lambda$ and $\nu_l\in I_{j_l}$. Note that k is finite because $f(\eta)$ is finitary.

Let $V=\bigcup_{l<k}\operatorname{orco}(\nu_l)$. Because the S_j are pairwise disjoint and $\delta\in S_i$, $\sup[\operatorname{orco}(\nu_l)\cap\delta]$ cannot be δ (even when $l(\nu_l)=\omega$). Hence $V\cap\delta$ is a bounded subset of δ, hence bounded by some $\delta_{n(0)}$.

Now let

$$A_n=\{\tau_1(\bar{c}_\nu): \text{ for some } \nu\in\{\eta_m{}^\wedge\langle\alpha_m\rangle, \eta_m{}^\wedge\langle\beta_m\rangle: m<n\},$$
$$f(\nu)=\tau_\nu(\bar{c}_\nu) \text{ and } \tau_1 \text{ is a subterm of } \tau_\nu\}.$$

So A_n increases with n. Let $B=\{\tau_1(\langle j_0,\nu_0\rangle,\cdots): \tau_1 \text{ a subterm of } \tau_\eta\}$. As $f(\eta)$ is strongly finitary, B is finite. Remember $<^*$ is a well ordering of $M^*(I_i^-)$, so there is $n(1)$ such that for every $b\in B$:

(1) $b\in\bigcup_{n<\omega}A_n\Rightarrow b\in A_{n(1)}$,

(2) $\min\{a\in\bigcup_{n<\omega}A_n: b\leqq^* a\}$ belong to $A_{n(1)}$, if there is such a.

Let $n>n(0)$, $n(1)$, now $\langle\eta_n{}^\wedge\langle\alpha_n\rangle,\eta\rangle$ and $\langle\eta_n{}^\wedge\langle\beta_n\rangle,\eta\rangle$ exemplify the requirement in the definition of strong ψ_{ptr}-unembeddability. We check two representative cases. Say $f(\eta_n{}^\wedge\langle\alpha_n\rangle)=\tau_1(\bar{c})$, $f(\eta_n{}^\wedge\langle\beta_n\rangle)=\tau_1(\bar{d})$. We get the same term as $\alpha_n E^f_{\eta_n}\beta_n$.

We want to show that $\langle\bar{c},\langle\langle\nu_0,j_0\rangle,\cdots,\langle\nu_{k-1},j_{k-1}\rangle\rangle\rangle=\langle\bar{c},\bar{e}\rangle$ realizes the same quantifier-free type as $\langle\bar{d},\bar{e}\rangle=\langle\bar{d},\langle\langle\nu_0,j_0\rangle,\cdots,\langle\nu_{k-1},j_{k-1}\rangle\rangle\rangle$, and that the $<^*$ condition holds. Suppose $c_i \lessdot \nu_s$. By the choice of C, $\operatorname{orco}(c_i)\subseteq\delta$. Hence for any $\zeta<l(c_i)$, $c_i(\zeta)=\nu_s(\zeta)<\delta_{n(0)}<\alpha_{n(0)}<\beta_{n(0)}\leqq\beta_{n-1}$, so $c_i(\zeta)=d_i(\zeta)$ as $\alpha_n E^f_{\eta_n}\beta_n$. Thus $c_i=d_i$, so $d_i\lessdot\nu_s$.

Suppose $\tau_2(\bar{c}) <^* \tau_3(\bar{e})$, τ_2 a subterm of τ_1 and τ_3 a subterm of τ_η. Thus $\tau_2(\bar{c}) \in A_{n+1}$ and $\tau_3(\bar{e}) \in B$. Suppose $\tau_3(\bar{e}) \leqq^* \tau_2(\bar{d})$.

Let $a \in \bigcup_{n<\omega} A_n$ be minimal such that $\tau_3(\bar{e}) \leqq^* a$, thus $a \in A_{n(1)}$. Now $\tau_2(\bar{c}) <^* a$ and $n(1) < n$, so $\alpha_n E^f_\eta \beta_n$ implies that $\tau_2(\bar{d}) <^* a$ but $\tau_3(\bar{e}) \leqq^* \tau_2(\bar{d})$, so this contradicts the minimality of a.

For the following see e.g. [13].

2.3. Claim. *For ω-sequences of ordinals η, ν let $\eta <_b \nu$ mean $\{n : \eta(n) < \nu(n)\}$ is cofinite. Note $<_b$ is a partial ordering.*

(1) *If η_i $(i < \delta)$ is $<_b$-increasing,* $\mathrm{cf}\,\delta > 2^{\aleph_0}$, *then it has a least upper bound η (i.e. $\nu <_b \eta$ iff $(\exists i < \delta)\ \nu <_b \eta_i$).*

(2) *If $\lambda = \Sigma_n \lambda_n$, $\lambda_n < \lambda_{n+1} < \lambda$, λ_n regular, $\lambda > 2^{\aleph_0}$, then there is a subsequence $\langle \lambda'_n : n < \omega \rangle$ of $\langle \lambda_n : n < \omega \rangle$, a regular $\lambda^* \geqq \lambda$ and $\eta_i \in \Pi \lambda'_n$ for $i < \lambda^*$, such that: $i < j \Rightarrow \eta_i <_b \eta_j$; for $\delta < \lambda^*$, if $\mathrm{cf}\,\delta > 2^{\aleph_0}$, η_δ is the least upper bound of $\langle \eta_i : i < \delta \rangle$ (for $<_b$) and $\langle \lambda'_n : n < \omega \rangle$ is the least upper bound of $\langle \eta_i : i < \lambda^* \rangle$, Note $\lambda^* \leqq \lambda^{\aleph_0}$.*

Proof. (1) We choose by induction on i an ω-sequence of ordinals ν_i such that

(a) for no $j < i$, $\nu_j \leqq_b \nu_i$,

(b) for no $j < \delta$, $\nu_i \leqq_b \eta_j$,

where we first set $\nu_0(m) = \sup\{\eta_i(m) : i < \delta\}$ for all $m \in \omega$. We cannot define ν_i for every $i < (2^{\aleph_0})^+$, for otherwise by the Erdös–Rado theorem, since by (a)

$$[(2^{\aleph_0})^+]^2 = \bigcup_{m \in \omega} \{\{i, j\} : i < j < (2^{\aleph_0})^+, \nu_j(m) < \nu_i(m)\}$$

we would get a descending ω-sequence of ordinals, contradiction.

So let ν_i be defined for $i < \alpha$ only, where $\alpha < (2^{\aleph_0})^+$. Let $A = \{\nu_i(l) : l < \omega, i < \alpha\}$, and for every $j < \delta$, let $\eta^*_j \in {}^\omega A$ be defined by $\eta^*_j(n) = \min\{\zeta \in A : \zeta \geqq \eta_j(n)\}$ for all $n < \omega$. This is possible by our choice of ν_0. Clearly $\eta_j \leqq_b \eta^*_j$, and there are just $\leqq (2^{\aleph_0})^{\aleph_0} < \mathrm{cf}\,\delta$ possible η^*_j, hence for some $\eta^*_j = \eta^*$ for unboundedly many $j < \delta$. We check that η^* is the required least upper bound.

Clearly if $(\exists i < \delta)\ \nu <_b \eta_i$, then $\nu <_b \eta^*$. Now suppose $(\forall i < \delta)\ \nu \not<_b \eta_i$. Then $(\forall i < \delta)\ \nu \not\leqq_b \eta_i$ so for some $i < \alpha$, $\nu_i \leqq_b \nu$. Suppose $\nu <_b \eta^*$ and $\eta^* = \eta^*_k$. Now $\forall n\ (\nu_i(n) < \eta^*(n)$ iff $\nu_i(n) < \eta_k(n))$. Hence, since $\nu_i \leqq_b \nu <_b \eta^*$ we have $\nu_i <_b \eta_k$, contradiction.

(2) Choose inductively on i, $\eta_i \in \Pi_{n<\omega} \lambda_n$ such that $\eta_j <_b \eta_i$ for $j < i$. As $\eta_i \neq \eta_j$ for $i \neq j$, it follows that for some $i < |\Pi_{n<\omega} \lambda_n|^+$, η_i cannot be defined; so suppose δ is the first such i. We can easily prove that δ is limit (otherwise

$\eta_{\delta-1} <_b \langle \eta_{\delta-1}(n)+1 : n < \omega \rangle$, contradiction) and of cofinality $> \lambda$ (otherwise $\operatorname{cf}\delta \leqq \lambda$ hence $\operatorname{cf}\delta < \lambda$ hence $\operatorname{cf}\delta < \lambda_n$ for some n; let $\delta = \bigcup_{\zeta<\operatorname{cf}\delta} i(\zeta)$ with $i(\zeta) < \delta$ for $\zeta < \operatorname{cf}\delta$, and let $\eta_\delta(m)$ be 0 for $m \leqq n$ and $\sup\{\eta_{i(\zeta)}(m)+1 : \zeta < \operatorname{cf}\delta\}$ for $m > n$).

So $\operatorname{cf}\delta > \lambda \geqq 2^{\aleph_0}$, so by 2.3(1) $\langle \eta_i : i < \delta \rangle$ has a least upper bound η^*. Clearly w.l.o.g. $\eta^*(n) \leqq \lambda_n$ for every n. If $\eta^* <_b \langle \lambda_n : n < \omega \rangle$ then modifiying η^* on a finite set we obtain a function which can serve as η_δ, contradiction to the choice of δ. Hence $A = \{n : \eta^*(n) = \lambda(n)\}$ is infinite; let it be $A = \langle n_k : k < \omega \rangle$, strictly increasing. Let $\lambda'_k = \lambda_{n_k}$ and $\eta'_i = \langle \eta_i(n_k) : k < \omega \rangle$. Clearly $\langle \eta'_i : i < \delta \rangle$ is $<_b$-increasing, and it is easily checked that $\langle \lambda'_k : k < \omega \rangle$ is the l.u.b. of $\langle \eta'_i : i < \delta \rangle$. Let $\lambda^* = \operatorname{cf}\delta$ and let $i(\zeta)$ $(\zeta < \lambda^*)$ be increasing, continuous and unbounded below δ. We define η''_ζ for $\zeta < \lambda^*$ as follows. For $\zeta = 0$, ζ a successor, or $\operatorname{cf}\zeta \leqq 2^{\aleph_0}$ let $\eta''_\zeta = \eta'_{i(\zeta)}$. For $\operatorname{cf}\zeta > 2^{\aleph_0}$ let η'''_ζ be the l.u.b. of η''_j for $j < \zeta$ (by 2.3(1)); for all $n < \omega$ let $\eta''_\zeta(n) = \min(\eta'''_\zeta(n), \eta'_{i(\zeta)}(n))$. Clearly λ^*, η''_i $(i < \lambda^*)$, λ'_n $(n < \omega)$ are as required.

2.4. Claim. *Suppose λ is singular, $\operatorname{cf}\lambda > \aleph_0$, f is a function from ${}^{\omega<}\lambda$ to finite subsets of ${}^{\omega\geqq}\lambda$ (or even subsets of ${}^{\omega\geqq}\lambda$ of power $< \operatorname{cf}\lambda$). Assume $\lambda = \Sigma_{i<\operatorname{cf}\lambda} \lambda_i$, λ_i strictly increasing and continuous for $i < \operatorname{cf}\lambda$; we suppose $\lambda_i = \Sigma_{n<\omega} \lambda_{i,n}$, where $\operatorname{cf}\lambda < \lambda_{i,0}$, $\forall n$ $(\lambda_{i,n} < \lambda_{i,n+1}$, $\lambda_{i,n} \in \{\lambda_j^+ : j < i\})$. Let $S = \{i < \operatorname{cf}\lambda : \operatorname{cf} i = \aleph_0\}$; recall that S is stationary.*

Then there is a closed unbounded $C \subseteq \operatorname{cf}\lambda$ such that for all $i \in C \cap S$ there is a T such that

(1) *$T \subseteq \bigcup_{n<\omega} \Pi_{m<n} \lambda_{i,m}$; $\langle\ \rangle \in T$, T closed under initial segments;*
(2) *$\forall \eta \in T (l(\eta) = n \Rightarrow |\{\alpha < \lambda_{i,n} : \eta^\frown\langle\alpha\rangle \in T\}| = \lambda_{i,n})$;*
(3) *$\forall \eta \in T$ $(f(\eta) \subseteq {}^{\omega\geqq}\lambda_i)$.*

Proof. For each $\eta \in {}^{\omega>}\lambda$ choose $g(\eta) < \operatorname{cf}\lambda$ so that $f(\eta) \subseteq \bigcup \{{}^{\omega\geqq}\zeta : \zeta < \lambda_{g(\eta)}\}$. Then instead of (3) we want

(3′) $\forall \eta \in T$ $(g(\eta) \leqq i)$.

Now we define a game G_i for each $i < \operatorname{cf}\lambda$ such that $\operatorname{cf} i = \aleph_0$: the game is of length ω, and in the nth move, player I chooses $A_n \subseteq \lambda_{i,n}$ with $|A_n| < \lambda_{i,n}$, and player II chooses $\eta_n \in \lambda_{i,n}$. Player II wins if $(l < n \Rightarrow \eta_l < \eta_n)$, $\eta_n \notin A_n$, and $g(\langle \eta_0, \cdots, \eta_n \rangle) \leqq i$; otherwise player I wins. Now

(4) *if $i < \operatorname{cf}\lambda$, $\operatorname{cf} i = \aleph_0$, $g(\langle\ \rangle) \leqq i$, and II has a winning strategy, then a desired T exists.*

For, let η be a winning strategy for II. Thus $\forall n \in \omega$, $\forall A \in {}^{n+1}\mathscr{P}\lambda$ such that $\forall m \leqq n$ $(A_m \in [\lambda_{i,m}]^{<\lambda_{i,m}})$ we have $\eta_A \in \lambda_{i,n}$, $\eta_{A\restriction(m+1)} < \eta_A$ for all $m < n$, $\eta_A \notin A_n$, and $g(\langle \eta_{A\restriction 1}, \cdots, \eta_{A\restriction(n+1)}\rangle) \leqq i$. Then $T = \{\langle \eta_{A\restriction 1}, \cdots, \eta_{A\restriction(n+1)}\rangle : \text{such } A\} \cup \{\langle\ \rangle\}$ is as desired. Thus we may assume

(5) $S' = \{i < \mathrm{cf}\,\lambda : \mathrm{cf}\, i = \omega$ *and* II *does not have a winning strategy for* $G_i\}$ *is stationary in* $\mathrm{cf}\,\lambda$.

Now the game G_i is open, so by the Gale–Stewart theorem is determined. Hence for each $i \in S'$ choose a winning strategy F_i for I. Thus

(6) $\forall n \in \omega$, $\forall \eta \in \Pi_{m<n} \lambda_{i,m}$ $(F_i(\eta) \in [\lambda_{i,n}]^{<\lambda_{i,n}})$;

(7) $\forall \eta \in \Pi_{m<\omega} \lambda_{i,m}$ *either* (a) $\exists l < n < \omega$ $(\eta_l \geqq \eta_n)$ *or* (b) $\exists n < \omega$, $\eta_n \in F_i(\eta \restriction n)$ *or* (c) $\exists n < \omega$, $g(\eta \restriction n) > i$.

Now choose a regular $\kappa > \aleph_0$ so that g, $\langle F_i : i \in S'\rangle$, $\langle \lambda_i^+ : i < \mathrm{cf}\,\lambda\rangle \in H(\kappa)$.

Remember $H(\kappa)$ is the family of sets with transitive closure of power $< \kappa$, and that $(H(\kappa), \in)$ is a model of ZFC$^-$. Let $<$ be a well-ordering of $H(\kappa)$.

For all $\delta < \mathrm{cf}\,\lambda$ let A_δ be the closure of $\delta \cup \{g, \langle F_i : i \in S'\rangle, \langle \lambda_i^+ : i < \mathrm{cf}\,\lambda'\rangle\}$ under Skolem functions within the structure $\langle H(\kappa), \in, <\ \rangle$. Then $C = \{\delta < \mathrm{cf}\,\lambda : A_\delta \cap \mathrm{cf}\,\lambda = \delta\}$ is closed unbounded in $\mathrm{cf}\,\lambda$. Thus there is $\delta \in S'$ and an elementary substructure $\langle N, \in, <\ \rangle$ of $\langle H(\kappa), \in, <\ \rangle$ such that $|N| < \mathrm{cf}\,\lambda$ and $N \cap \mathrm{cf}(\lambda) = \delta$, with g, $\langle F_i : i \in S'\rangle$, $\langle \lambda_i^+ : i < \mathrm{cf}\,\lambda\rangle \in N$, clearly $\lambda_i^+ \in N$ iff $i < \delta$, hence $\lambda_{\delta,m}$ belongs to N for each m. However $\delta \not\in N$, hence $\{\lambda_{\delta,m} : m < \omega\} \not\in N$.

Now we define $\eta \in \Pi_{m\in\omega} \lambda_{\delta,m}$ so as to contradict (7). Suppose $\eta_m \in N$ constructed for all $m < n$. Using elementarity and absoluteness of suitable formulas we see that the set

$$A^* = \bigcup \{F_j(\langle \eta_0, \cdots, \eta_{n-1}\rangle) : j < \mathrm{cf}\,\lambda, \langle \eta_0, \cdots, \eta_{n-1}\rangle \in \mathrm{Dom}\, F_j, j \in S',$$

$$\lambda_{j,0} = \lambda_{\delta,0}, \cdots, \lambda_{j,n-1} = \lambda_{\delta,n-1}, \lambda_{j,n} = \lambda_{\delta,n}\}$$

has power $< \lambda_{\delta,n}$ and is in N. Since $\exists \alpha (\eta_{n-1} < \alpha < \lambda_{\delta,n} \wedge \alpha \not\in A^*)$ holds in $\langle H(\kappa), \in, <\ \rangle$ it holds in $(N, \in, <)$ and this gives η_n. this completes the construction, and it is easily seen that (7) is contradicted.

Remarks. (1) Rubin and Shelah [9] deal with such theorems for their own sake.

(2) We can get in this direction more results. If $2^{\mathrm{cf}\lambda} < \lambda$, λ_{i+1} regular, then we can find a closed unbounded set $\{\alpha(i) : i < \mathrm{cf}\,\lambda\}$, $\alpha(i+1)$ successor and $T \subseteq {}^{\omega>}\lambda$, such that: $\langle\ \rangle \in T$, $\eta \in T$, $\mathrm{Max}[\mathrm{orco}(\eta)] < \lambda_{i+1} < \lambda_j$ implies $\{\alpha < \lambda_j : \eta^\wedge\langle\alpha\rangle \in T\}$ has power λ_j, and implies also $g(\eta) < j$.

(3) In (2) we can replace "$2^{\mathrm{cf}\lambda} < \lambda$" by "there is a family S of closed

unbounded subsets of cf λ such that $|S| < \lambda$, and every closed unbounded subset of cf λ contains one of them."

On the other hand, if $\mu = \mu^{<\mu}$ in V let us add $\lambda > \mu$ generic closed unbounded subsets of μ (by $Q = \{f : \operatorname{Dom} f$ a subset of λ of power $< \mu$, $f(i)$ the characteristic function of a closed bounded subset of $\mu\}$, the characteristic function of C_i is $\bigcup \{f(i) : f$ in the generic set$\}$).

Let $\{C_\eta : \eta \in {}^{\omega>}\lambda\}$ be another enumeration of $\{C_i : i < \lambda\}$, and define g:

$$g(\eta^\wedge\langle\alpha\rangle) = \min\{i < \operatorname{cf}\lambda : i \in C_\eta,\ \lambda_i > \alpha\}.$$

Clearly for this g the conclusion of remark (2) fails.

(4) We can generalize Claim 2.4 as in [16], [17], by attaching a filter for each node, i.e.,

2.5. CLAIM. *In 2.4, suppose that for $\eta \in {}^{\omega>}\lambda$, $D_{\eta,\mu}$ is a $(\operatorname{cf}\lambda)^+$-complete filter on $\{\eta^\wedge\langle\alpha\rangle : \alpha < \mu\}$ which include $\{\eta^\wedge\langle\alpha\rangle : \alpha_0 < \alpha < \mu\}$ for each $\alpha_0 < \mu$, then in the conclusion we can demand*:

(C) *for $\eta \in T$, $\{\eta^\wedge\langle\alpha\rangle : \ \eta^\wedge\langle\alpha\rangle \in T, \alpha < \lambda_{i,n}\} \neq \emptyset \bmod D_{\eta,\lambda_{i,n}}$.*

The proof is the same, only in the definition of the games G_i, instead of $|A_n| < \lambda_{i,n}$ we demand $A_n = \emptyset \bmod D_{\eta_{n-1},\lambda_{i,n}}$.

Of course, the fact that we use the tree ${}^{\omega>}\lambda$ is just for notational convenience.

An example of such a system of filters is that for

$$I = \{\langle\langle\alpha_0,\beta_0\rangle,\cdots,\langle\alpha_{n-1},\beta_{n-1}\rangle\rangle : \alpha_l, \beta_l < \lambda \text{ for } l < n, n < \omega\}$$

it is natural to define for $\eta \in I$, $\mu < \lambda$ a filter $D_{\eta,\mu}$ as the filter on $\{\eta^\wedge\langle\langle\alpha,\beta\rangle\rangle : \alpha,\beta < \lambda\}$ generated by the sets $\{\eta^\wedge\langle\langle\alpha,\beta\rangle\rangle : \alpha,\beta < \lambda, \alpha E \beta\}$ for any equivalence relation E on λ with $< \mu$ equivalence classes. If μ is regular, $\chi < \mu \Rightarrow \chi^{<\kappa} < \mu$ then $D_{\eta,\mu}$ is κ-complete.

2.6. THEOREM. *Suppose λ is singular, $\lambda^{\aleph_0} = \lambda$. Then for any μ, $\mu^{\aleph_0} < \lambda$,*

(1) *K^ω_{tr} has the full strong $(\lambda,\lambda,\mu,\aleph_0)$-$\psi_{\mathrm{tr}}$-bigness property,*

(2) *K^ω_{ptr} has the full strong $(\lambda,\lambda,\mu,\aleph_0)$-$\psi_{\mathrm{ptr}}$-bigness property.*

(3) *Suppose in addition $\mu^{<\kappa} < \lambda$, κ regular, then in both cases we can replace $\aleph_0$ by κ, if we add "for f which are strongly finitary on P_ω".*

We concentrate, usually, on (1). For the changes we need for (2) see the proof of 2.8. We could have used the partition theorem on $I \in K^\omega_{\mathrm{ptr}}$ but prefer another way.

Case A. $\lambda = 2^\chi = \chi^{\aleph_0}$; $\mu^{\aleph_0} < \lambda$, $\lambda > 2^{\aleph_0}$

PROOF. (1) w.l.o.g. $(\forall \chi_1 < \chi)\ \chi_1^{\aleph_0} < \chi$, $\chi = \sum_{n<\omega}\chi_n$, χ_n a successor, $\chi_n^- = (\chi_n^-)^{\aleph_0}$, $\mu < \chi_0$.

We let $I^0=\bigcup_{n<\omega}\prod_{i<n}\chi_i$, $I^1=I^0\cup\prod_{l<\omega}\chi_l$, and we shall choose I_α $(\alpha<\lambda)$, $I^0\subseteq I_\alpha\subseteq I^1$ such that $\langle I_\alpha:\alpha<\lambda\rangle$ will exemplify our conclusion. For this we have to deal with all pairs $\alpha<\lambda$, $F:I_\alpha\to M^*_{\mu,\aleph_0}(\Sigma_{\beta<\lambda,\beta\neq\alpha}I_\beta)$; there are too many such pairs, however the number of pairs $\langle\alpha,F\restriction I^0\rangle$ is $\leqq\lambda^\chi=\lambda$, so let $\langle\langle\alpha_\zeta,F_\zeta\rangle:\zeta<\lambda\rangle$ be an enumeration of all such possible pairs each appearing λ times. Define by induction on $\zeta<\lambda$, $I_{\alpha,\zeta}$ $(\alpha<\lambda)$, $J_{\alpha,\zeta}$ $(\alpha<\lambda)$ such that:

(a) $I_{\alpha,\zeta}$, $J_{\alpha,\zeta}$ are disjoint subsets of I^1-I^0 increasing with ζ,

(b) $\Sigma_\alpha(|I_{\alpha,\zeta}|+|J_{\alpha,\zeta}|)\leqq\chi+|\zeta|$,

(c) if $I^0\subseteq I'_\alpha\subseteq I^1$, $I_{\alpha,\zeta}\subseteq I'_\alpha$, $J_{\alpha,\zeta}\cap I'_\alpha=\emptyset$ for $\alpha<\lambda$ and $F:I'_{\alpha_\zeta}\to M^*_{\mu,\aleph_0}(\Sigma_{\beta<\lambda,\beta\neq\alpha_\zeta}I'_\beta)$,
F extends F_ζ then the conclusion of Definition 1.2 is satisfied (for K^ω_{tr}, $\psi=\psi_{\mathrm{tr}}$).

Clearly it is enough to carry the induction (and then $I_\alpha=I^0\cup\bigcup_{\zeta<\lambda}I_{\alpha,\zeta}$ $(\alpha<\lambda)$ exemplify the conclusion).

So let us do the ζ-th step (so we suppress ζ as an index). Let

$$I^+=\sum_{\substack{\beta<\lambda\\ \beta\neq\alpha_\zeta}}\bigcup_{\xi<\zeta}I_{\beta,\xi},\qquad I^*=\bigcup_{\substack{\beta<\lambda\\ \beta\neq\alpha_\zeta}}\bigcup_{\xi<\zeta}I_{\beta,\zeta}\cup\{\langle\beta\rangle:\beta<\lambda\}$$

(so I^+ universe will be $\bigcup_{\beta<\lambda,\beta\neq\alpha_\zeta}(\bigcup_{\xi<\zeta}I_{\beta,\xi}\times\{\beta\})$); w.l.o.g. F_ζ is into $M^*_{\mu,\aleph_0}(I^+)$. We let $\langle\eta,\alpha\rangle\restriction l=\langle\eta\restriction l,\alpha\rangle$.

Remember that the universe of $I^+=\Sigma_{\beta<\lambda,\beta\neq\alpha_\zeta}\bigcup_{\xi<\zeta}I_{\beta,\xi}$ is $\cup\{I_{\beta,\xi}\times\{\beta\}:\xi<\zeta$ and $\beta<\lambda$, $\beta\neq\alpha_\zeta\}$. Let, for $\langle\eta,\beta\rangle\in I^+$, $\langle\eta,\beta\rangle\restriction k=\langle\eta\restriction k,\beta\rangle$ and $l(\langle\eta,\beta\rangle)=l(\eta)$. Let $F(\eta)$ be the set of $\nu\in I^+$ "appearing" in $F_\zeta(\eta)$. Call a $T\subseteq I^0$ *big* if T^0 is closed under initial segments, $\langle\ \rangle\in T$ and for every $\eta\in T$, $l(\eta)=k$, $\{i:\eta\hat{}\langle i\rangle\in T\}$ has power χ_k. By Rubin–Shelah [9] theorem 4.9 there is a big $T\subseteq I^0$ and A_η for each $\eta\in T$ such that

(1) A_η is a countable subset of I^+, which includes $F(\eta)$,

(2) if $\nu=\eta_1\restriction k=\eta_2\restriction k$, $\eta_1(k)\neq\eta_2(k)$, $\eta_1\in T$ and $\eta_2\in T$, then $A_{\eta_1}\cap A_{\eta_2}=A_\nu$.

We can easily find T', B_η $(\eta\in T')$ such that:

(α) $T'\subseteq T$ is big,

(β) for every $\nu=\eta\hat{}\langle i\rangle\in T'$ there is ρ_ν satisfying:

($\beta1$) $\rho_\nu=\eta\hat{}\langle j_\nu\rangle\in T$, $j_\nu<i$,

($\beta2$) $\rho_\nu\notin T'$, but $(\forall j<i)\ [\eta\hat{}\langle j\rangle\in T'\to j<j_\nu]$,

($\beta3$) $f(\nu)$, $f(\rho_\nu)$ realizes the same quantifier free type over B_η in I^+,

($\beta4$) $B_\nu=B_\eta\cup A_\nu\cup A_{\rho_\nu}$, B_ν countable closed under initial segments.

The definition of $T'_n=T'\cap(\bigcup_{m\leqq n}\prod_{l<m}\chi_l)$ and B_ν $(\eta\in T'_n)$ is done by induction on n.

Clearly

(*) if $\eta = \nu_1 \restriction l = \nu_2 \restriction l$, $\nu_1 \in T'$, $\nu_2 \in T'$, $\nu_1(l) \neq \nu_2(l)$ then $B_{\nu_1} \cap B_{\nu_2} = B_\eta$.

Hence for all $\eta \in \operatorname{Lim} T'(=\{\nu \in I^1 - I^0 : (\forall l)\ \nu \restriction l \in T'\}$ except) $\leqq |\zeta| + \chi$ many, the following holds:

$$(**)\qquad \text{for every } \rho \in I^+,\quad l(\rho) = \omega \quad \text{for some } k, m < \omega,$$
$$\rho \restriction k \in B_{\eta \restriction m}, \qquad \rho \restriction (k+1) \notin \bigcup_{l<\omega} B_{\eta \restriction l}.$$

Let $\eta_\zeta \in \operatorname{Lim} T'$ satisfy (**) and we define

$$I_{\beta,\zeta} = \begin{cases} \bigcup_{\xi<\zeta} I_{\beta,\xi} \cup \{\eta_\zeta\} & \beta = \alpha_\zeta, \\ \bigcup_{\xi<\zeta} I_{\beta,\xi} & \text{otherwise.} \end{cases}$$

$J_{\beta,\zeta} = \bigcup_{\xi<\zeta} J_{\beta,\xi} \cup \{\eta : \eta \in I^1 - I^0$, for every $k < \omega \langle \eta \restriction k, \beta \rangle \in \bigcup_{l<\omega} B_{\eta_\zeta \restriction l}$ but $\eta \notin I_{\beta,\zeta}\}$.

Let us prove we have defined $I_{\beta,\zeta}$, $J_{\beta,\zeta}$ $(\beta < \lambda)$ as required. Now (a), (b) are easy, so assume $I^0 \subseteq I'_\alpha \subseteq I^1$, $I_{\alpha,\zeta} \subseteq I'_\alpha$, $J_{\alpha,\zeta} \cap I'_\alpha = \emptyset$, F^* extend F_ζ, $F^* : I'_{\alpha_\zeta} \to M^*_{\mu,\aleph_0}(\Sigma_{\beta<\lambda,\beta\neq\alpha_\zeta} I'_\beta)$. Now $F^*(\eta_\zeta)$ is $\tau(\nu_0, \cdots, \nu_{n-1})$, $\nu_l \in I^+$, n is finite.

We can assume (letting $\eta^* = \eta_\zeta$) $\nu_l \in \bigcup_m B_{\eta_\zeta \restriction m}$ iff $l \geqq n(0)$. By (**) for each $l < n(0)$ for some $k(l) < \omega$.

$$\nu_l \restriction k(l) \in \bigcup_m B_{\eta^* \restriction m}, \qquad \nu_l \restriction (k(l)+1) \notin \bigcup_m B_{\eta^* \restriction m}.$$

For $m < \omega$ let $B_m = B_{\eta^* \restriction m}$. Choose $m(*) < \omega$ large enough so that $\nu_l \in B_{m(*)}$ for $l \geqq n(0)$, $\nu_l \restriction k(l) \in B_{m(*)}$ for $l < n(0)$, and for $l < n(0)$

$$\begin{aligned} &\operatorname{Min}\{\alpha : \nu_l \restriction k(l)^\frown\langle\alpha\rangle \in B_{m(*)}, \alpha \geqq \nu_l(k(l))\} \\ &= \operatorname{Min}\{\alpha : \nu_l \restriction k(l)^\frown\langle\alpha\rangle \in \bigcup_m B_m, \alpha \geqq \nu_l(k(l))\} \end{aligned}$$

(or both are undefined).

Now $\langle \eta^*, \eta^* \restriction m(*) \rangle$, $\langle \eta, \rho_{\eta^* \restriction m(*)} \rangle$ are as required.

Case B. $\lambda = \lambda^{\aleph_0}$ is strong limit singular

Proof. Choose λ_i $(i < \operatorname{cf} \lambda)$ such that:

(a) λ_i is increasing continuous with limit λ, $(2^\mu)^+ = \lambda_0$,

(b) λ_δ is strong limit for limit $\delta < \operatorname{cf} \lambda$,

(c) λ_{i+1} is a successor cardinal moreover $\chi^{\lambda_i} < \lambda_{i+1}$ for $\chi < \lambda_i$.

For each limit $\delta < \operatorname{cf} \lambda$, $\operatorname{cf} \delta = \aleph_0$ choose $\lambda_{\delta,n} \in \{\lambda_{j+1} : j < \delta\}$, $\lambda_\delta = \Sigma_{n<\omega} \lambda_{\delta,n}$, $\lambda_{\delta,n} < \lambda_{\delta,n+1}$.

We shall choose $I_{\delta,\alpha}$ ($\delta < \mathrm{cf}\,\lambda$ limit, $\mathrm{cf}\,\delta = \aleph_0$, $\alpha < \lambda_\delta$), $I_{\delta,\alpha} \subseteq \Pi_{n<\omega}\lambda_{\delta,n}$, and let $\delta < \mathrm{cf}\,\lambda$ limit, $\lambda_\delta > \alpha\}$ such that:

(*) if $\delta < \mathrm{cf}\,\lambda$ is limit, $\mathrm{cf}\,\delta = \aleph_0$, $\alpha < \lambda_\delta$, $F : I_\alpha \to M^*_{\mu,\aleph_0}(\Sigma_{\beta<\lambda_\beta,\beta\neq\alpha} J)$ and some big $T \subseteq \bigcup_{n<\omega}\Pi_{l<n}\lambda_{\delta,l}$, for every $\eta \in T$,

$$F(\eta) \in M^*_{\mu,\aleph_0}(\Sigma_{\beta<\lambda_\delta,\beta\neq\alpha}(I_\beta \cap \bigcup_{i<\delta} {}^{\omega\geqq}\lambda_i)),$$

then the conclusion of Definition 1.2 is satisfied.

This is enough by 2.4 and the proof that it can be satisfied is like that of Case A.

Case C. $\mu \leqq \chi^{\aleph_0} < \lambda \leqq 2^\chi$

The proof is by 2.7 (for 2.6(1), (3)) and 2.8 (for 2.6(2), (3)).

2.7. THEOREM. *Suppose $\lambda^{\aleph_0} = \lambda$, $\mu \leqq \lambda$, Then*

(1) *K^ω_{tr} has the full strong $(2^\lambda, \lambda^+, \mu, \aleph_0)$-$\psi_{\mathrm{tr}}$-bigness property.*

(2) *If in addition $\lambda^{<\kappa} \leqq \lambda$, κ regular, then we can replace $\aleph_0$ by κ provided that we add "for f which are strongly finitary on P_ω".*

REMARK. The interesting consequence is when we replace both 2^λ and λ^+ by some χ, $\lambda^+ \leqq \chi \leqq 2^\lambda$.

PROOF. Let $S^\alpha \subseteq \{\delta < \lambda^+ : \mathrm{cf}\,\delta = \aleph_0\}$ (for $\alpha < \lambda$) be pairwise disjoint stationary subsets of λ^+. Let A_i be a subset of λ for $i < 2^\lambda$ such that no one is included in a union of countably many others (in (2) — less than κ many); such A_i ($i < 2^\lambda$) exists by Engelking and Karlowicz [4].

Let $S_i = \bigcup_{\alpha\in A_i} S^\alpha$, $I_i = {}^{\omega>}(\lambda^+) \cup \{\eta : \eta$ an increasing ω-sequence of ordinals $< \lambda^+$ whose limit belongs to $S_i\}$.

Suppose $f : I_i \to M_{\mu,\kappa}(\Sigma_{j\neq i} I_j)$, and let $f(\eta) = \tau_n(\langle \nu_{\eta,0}, j_{\eta,0}\rangle, \langle \nu_{n,1}, j_{n,1}\rangle, \cdots)$, $\nu_{\eta,l} \in I_{j_{\eta,l}}$. For any $\eta \in {}^\omega(\lambda^+)$ let $F(\eta) = \{j :$ for some $n < \omega$, and l, $j = j_{\eta\restriction n,l}\}$. Clearly $F(\eta)$ is a subset of $2^\lambda - \{i\}$ of power $\leqq \aleph_0$ (or $< \kappa$ in (3)), hence $A_i \not\subseteq \bigcup_{j\in F(\eta)} A_j$, so there is $\alpha < \lambda$ such that $\alpha \in A_i - \bigcup_{j\in F(\eta)} A_j$. Now for each $\alpha < \lambda$, $W_\alpha = \{\eta \in {}^\omega(\lambda^+) : \alpha \notin A_j$ for $j \in F(\eta)\}$ is a closed set, and by the previous sentence $\bigcup_{\alpha\in A_i} W_\alpha = {}^\omega(\lambda^+)$. Hence by 1.2 of Rubin and Shelah [9] lemma 2.14, there are α and $T \subseteq {}^{\omega>}(\lambda^+)$ such that:

(a) $\langle\ \rangle \in T$,

(b) $\eta \in T$ implies that for λ^+ ordinals $\gamma < \lambda^+$, $\eta^\frown\langle\gamma\rangle \in T$,

(c) for every $\eta \in T$, $j = j_{\eta\restriction m,l}$: α does not belong to A_j.

Now we can work with T, and get a contradiction as in 2.1 by restricting f to $I'_i = ({}^{\omega>}(\lambda^+) \cap T) \cup \{\eta \in {}^\omega(\lambda^+) \cap I_i$: for every $n < \omega$, $\eta \restriction n \in T\}$.

2.8. THEOREM. *Suppose* $\mu \leqq \lambda = \lambda^{\aleph_0}$, *then*

(1) $K^{\omega}_{\mathrm{ptr}}$ *has the full strong* $(2^{\lambda}, \lambda^+, \mu, \aleph_0)$-$\psi_{\mathrm{ptr}}$-*bigness property.*

(2) *If in addition* $\lambda^{<\kappa} \leqq \lambda$, κ *regular, then* $K^{\omega}_{\mathrm{ptr}}$ *has the full strong* $(2^{\lambda}, \lambda^+, \mu, \kappa)$-*bigness property for* f *which are strongly finitary on* P_{ω}.

PROOF. (1) Define S^{α}, S_i, A_i as in the proof of 2.7 and

$$I_i = \{\langle\langle\alpha_0, \beta_0\rangle, \cdots, \langle\alpha_n, \beta_n\rangle, \alpha_{n+1}\rangle : n < \omega, \alpha_l < \beta_l < \lambda^+\}$$
$$\cup \{\langle\langle\alpha_0, \beta_0\rangle, \langle\alpha_1, \beta_1\rangle, \cdots, \langle\alpha_n, \beta_n\rangle, \cdots\rangle : n < \omega, \alpha_l < \beta_l < \alpha_{l+1} < \lambda^+$$
$$\text{for } l < \omega \text{ and } \bigcup_{n<\omega} \alpha_n \text{ belong to } S_i\}$$

Suppose $f : I_i \to M^*_{\mu,\kappa}(\Sigma_{j \neq i} I_j)$, let χ be big enough, N^* be an expansion of $(H(\chi), \in)$ by Skolem functions, and individual constants for κ, μ, λ, i, $\langle I_j : j < 2^{\lambda}\rangle$, $\langle S_j : j < 2^{\lambda}\rangle$, $\langle S^{\alpha} : \alpha < \lambda\rangle$.

For $\eta \in {}^{\omega \geqq}(2^{\lambda})$, let N_{η} be the Skolem hull of $\{\eta(l), N_{\eta \restriction l} : l < l(\eta)\}$ (in N^*), hence if $l(\eta) = \omega$, $N_{\eta} = \bigcup_{l<\omega} N_{\eta \restriction l}$ and $N_{\eta \restriction l} \in N_{\eta \restriction l+1}$.

For $\eta \in {}^{\omega}(\lambda^+)$, let $F(\eta) = N_{\eta} \cap 2^{\lambda} - \{i\}$, so $F(\eta)$ is a countable subset of $2^{\lambda} - \{i\}$, hence $A_i \not\subseteq \bigcup_{j \in F(\eta)} A_j$; let $W_{\alpha} = \{\eta \in {}^{\omega}(\lambda^+) : \alpha \notin A_j$ for $j \in F(\eta)\}$, and there are α, T as in the proof of 2.7:

(a) $\langle\ \rangle \in T$, $\alpha \in A_i$,

(b) $\eta \in T$ implies that for λ^+ ordinals, $\gamma < \lambda^+$, $\eta^\frown\langle\gamma\rangle \in T$,

(c) for $\eta \in T$, *if* $j \in N_{\eta}$, $j < 2^{\lambda}$, $j \neq i$, then α does not belong to A_j.

Now we choose $\eta \in {}^{\omega}(\lambda^+)$ such that $\eta(n) > \sup(N_{\eta \restriction n} \cap \lambda^+)$ and $\bigcup_{n<\omega} \eta(n) \in S_{\alpha}$; this is possible by (b) and as S_{α} is stationary, $[\delta \in S_{\alpha} \Rightarrow \operatorname{cf} \delta = \aleph_0]$. Now as in the proof of 2.2, we define by induction on n, α_n, β_n; $\eta(n) < \alpha_n < \beta_n \in N_{\eta \restriction (n+1)}$ and $\langle\langle\alpha_0, \beta_0\rangle, \cdots, \langle\alpha_n, \beta_n\rangle, \cdots\rangle$ will exemplify what we need.

2.9. THEOREM. (1) *Suppose* $\mu \leqq \lambda$, $\{A_i : i < \chi\}$ *is a family of subsets of* λ, *no one included in the union of* $\aleph_0$ *others, then* K^{ω}_{tr} *has the full strong* $(\chi, \lambda^{\aleph_0} + \lambda^+, \mu, \aleph_0)$-$\psi_{\mathrm{tr}}$-*bigness property.*

(2) *The parallels of* 2.7(2), 2.8 (*in the sense* 2.9(1) *is parallel to* 2.7(1)) *hold.*

PROOF. The same proof, using the A_i's above.

§3. Applications to Boolean algebras

3.1. DEFINITION. (1) For $I \in K^{\omega}_{\mathrm{tr}}$ let $B_{\mathrm{tr}}(J)$ be the Boolean algebra generated freely by x_{η} $(\eta \in I)$ except that $\eta \lessdot \nu \Leftrightarrow x_{\eta} > x_{\nu}$.

(2) For $I \in K^{\omega}_{\mathrm{ptr}}$ let $B_{\mathrm{ptr}}(I)$ be the Boolean algebra generated freely by x_η ($\eta \in I$) except that for $\eta \in I$, $l(\eta) = \omega$, $n < \omega$, $\eta = \langle\langle\alpha_0, \beta_0\rangle, \cdots, \langle\alpha_n, \beta_n\rangle \cdots\rangle$, the following holds:

$$x_\eta < x_{\eta\restriction n \frown \langle\alpha_n\rangle} \wedge x_\eta \cap x_{\eta\restriction n \frown \langle\beta_n\rangle} = 0.$$

(3) For $I \in K^{\omega}_{\mathrm{tr}}$ such that every $\eta \in I$, which has an immediate successor, has infinitely many immediate successors, let $B_{\mathrm{trr}}(I)$ be the Boolean algebra generated freely by x_η ($\eta \in I$) except that $x_{\eta\frown\langle\alpha\rangle} \cap x_{\eta\frown\langle\beta\rangle} = 0$ for $\alpha \neq \beta$ and $x_\eta < x_\nu$ for $\nu \lessdot \eta$.

3.2. Notation. We let x stand for tr or ptr or trr.

3.3. Definition. For Boolean algebras B, B_1 and $a^* \in B_1$ we define the "B-surgery of B_1 at a^*" or "surgery of B_1 at a^* by B", B_2, as a Boolean algebra extending B, $B_2 = [B_1 \restriction (-a^*)] \times [(B_1 \restriction a^*) * B]$ where $\times$ is a direct product $*$ free product. Alternatively B_2 is generated as follows: first make B disjoint to B_1 (by taking an isomorphic copy) and then B_2 is generated freely by $B_1 \cup B$ except the relations

$$0_{B_1} = 0_B,$$

$$a \cap b = c \quad (\text{for } a, b, c \in B_1, a \cap b = c \text{ in } B_1) \quad \text{similarly for } \cup,$$

$$1_{B_1} - b = c \quad (\text{for } b, c \in B_1, 1_{B_1} - b = c \text{ in } B_1),$$

$$a \cap b = c \quad (\text{for } a, b, c \in B, a \cap b = c) \quad \text{similarly for } \cup,$$

$$a < a^* \quad (\text{i.e. } a \cap a^* = a) \quad (\text{for } a \in B).$$

3.4. Construction. Let $x \in \{\mathrm{tr}, \mathrm{ptr}\}$, λ a cardinal. The idea is to construct a Boolean algebra by defining an increasing continuous sequence B_i ($i < \alpha$), B_0 trivial and we get B_{i+1} by a surgery of B_i at $a^*_i \in B_i$ by $B^*_i = B_x(I_i)$, $|I_i| = \lambda$, $I_i \in K^{\omega}_x$; where I_i is fully ψ_x-unembeddable into $\Sigma_{j \neq i} I_j$. We denote $B = \bigcup_{i<\alpha} B_i$ by $\mathrm{Sur}\langle I_i, a^*_i : i < \alpha\rangle$. Usually we want $\bigcup_{i<\lambda} B_i - \{0\} = \{a^*_i : i < \alpha\}$.

Definition. (1) A B.A. satisfies the λ-chain condition if there are no λ-elements which form an antichain (i.e. they are $\neq 0$, the intersection of any two is zero).

(2) A B.A. satisfies the strong λ-chain condition if among any λ-elements there are λ which are pairwise not disjoint.

3.5. Claim. *Let $x \in \{\mathrm{tr}, \mathrm{ptr}\}$, $I \in K^{\omega}_x$, λ uncountable regular.*

(i) *If $x = \mathrm{tr}$, then $B_x(I)$ satisfies the strong λ-chain condition.*

(ii) *If $x = \mathrm{ptr}$, then $B_x(I)$ satisfies the strong $(2^{\aleph_0})^+$-chain condition.*

PROOF. (i) First we take $x = \mathrm{tr}$ and check the strong λ-chain condition. Note

(1) $$x_{\eta_1} \cap \cdots \cap x_{\eta_k} \cap (-x_{\nu_1}) \cap \cdots \cap (-x_{\nu_h}) = 0 \quad \text{iff } \exists i, j \ (\nu_i \trianglelefteq \eta_j).$$

Now for $F \in [I]^{<\omega}$ let $x_F = \Pi_{\eta \in F} x_\eta$ and $\bar{x}_F = \Pi_{\eta \in F}(-x_\eta)$. To check the strong λ-chain condition it suffices to take $\Gamma \subseteq [I]^{<\omega} \times [I]^{<\omega}$ with $|\Gamma| = \lambda$ and $\forall (F, G) \in \Gamma \ (x_F \cap \bar{x}_G \neq 0)$ and find $\Gamma' \in [\Gamma]^\lambda$ s.t. $\forall (F, G), (F', G') \in \Gamma'$, $x_F \cap \bar{x}_G \cap x_{F'} \cap \bar{x}_{G'} \neq 0$. We may assume that $\langle F : (F, G) \in \Gamma \rangle$ and $\langle G : (F, G) \in \Gamma \rangle$ are Δ-systems, say with kernels K_1, K_2 resp. We may assume $(F, G) \neq (F', G') \Rightarrow F \neq F'$ and $G \neq G'$. We may assume $\exists m, n \in \omega \ \forall (F, G) \in \Gamma$ $(|F| = m \wedge |G| = n)$. Say $\eta_F : m \to F$, $\eta'_G : n \to G$ (are one-to-one, onto). We may assume $\forall j < n$, $\forall (F, G), (F', G') \in \Gamma$ (length $\eta_G(j) =$ length $\eta_G(j)$). Clearly then, using the Δ-system assumption,

(2) $\forall (F, G) \in \Gamma \ \forall i < m \ \forall j < n$ there is at most one $(F', G') \in \Gamma$ s.t. $\eta_{G'}(j) \trianglelefteq \eta_F(i)$.

Now

(3) $\forall i < m \ \forall j < n \ \forall \Gamma' \in [\Gamma]^\lambda \ \exists \Gamma'' \in [\Gamma']^\lambda \ \forall (F, G), (F', G') \in \Gamma'' \ (\eta'_G(j) \ntrianglelefteq \eta_F(i))$.

For, let $<^1$ be a well-order of Γ',

$$J_0 = \{\{(F, G), (F', G')\} \in [\Gamma']^2 : (F, G) <^1 (F', G') \text{ and } \eta_G(j) \trianglelefteq \eta_{F'}(i)\}.$$

J_1 similarly with ${}^1\!>$ in place of $<^1$. Then $[\Gamma']^2 = J_0 \cup J_1 \cup ([\Gamma']^2 - (J_0 \cup J_1))$, and $\lambda \to (\omega, \omega, \lambda)$ yields (3), using (2).

Applying (3) $m \cdot n$ times gives the desired result, by (1).

(ii) The case $x = \mathrm{ptr}$ is similar, but more complicated. First note

(4) $$x_{\eta_0} \cap \cdots \cap x_{\eta_{m-1}} \cap (-x_{\nu_0}) \cap \cdots \cap (-x_{\nu_{n-1}}) = 0$$

iff one of the following conditions holds:

(a) $\exists i, j < m \ (l(\eta_i) = \omega$ and $\mathrm{Suc}_R(\eta_j, \eta_i))$,

(b) $\exists i < m, \ \exists j < n \ (l(\eta_i) = \omega$ and $\mathrm{Suc}_L(\nu_j, \eta_i))$,

(c) $\exists i, j < m \ (l(\eta_i) = l(\eta_j) = \omega$ and $\exists p < \omega \ (\eta_i \restriction p$ and $\exists \alpha, \beta, \gamma \ [\eta_i(p) = \omega\langle \alpha, \beta \rangle$, and $\eta_j(p) = \langle \beta, \gamma \rangle)]))$.

Again we take $\Gamma \subseteq [I]^{<\omega} \times [I]^{<\omega}$ with $|\Gamma| = (2^{\aleph_0})^+$ and $\forall (F, G) \in \Gamma \ (x_F \cap \bar{x}_G \neq 0)$. We get Δ-systems as before. We may assume $(F, G) \neq (F', G') \Rightarrow F \neq F'$ (but possibly $G = G'$ always). We get m, n, η_F, η'_G as before, and again make the length assumption. Now

(5) $$\forall (F, G) \in \Gamma \ \forall i < m \ \forall j < n \text{ there is at most one } (F', G') \in \Gamma \text{ s.t. } \mathrm{Suc}_L(\eta_{G'}(j), \eta_F(i)).$$

Hence as before we may assume

$$(6)\qquad \forall i < m\ \forall j < n\ \forall (F,G),(F',G') \in \Gamma(\neg \operatorname{Suc}_L(\eta_{G'}(j), \eta_F(i))).$$

Similarly we may assume

$$(7)\qquad \forall i < m\ \forall j < n\ \forall (F,G),(F',G') \in \Gamma(\neg \operatorname{Suc}_R(\eta_{F'}(j), \eta_F(i))).$$

Now if $\sigma, \tau \in I$ with $l(\sigma) = l(\tau) = \omega$, and if $p < \omega$, we define $\sigma R_p \tau$ iff $\sigma \restriction p = \tau \restriction p$ and $\exists \alpha, \beta, \gamma\ (\sigma(p) = \langle \alpha, \beta \rangle$ and $\tau(p) = \langle \beta, \gamma \rangle)$. Now

$$(8)\qquad \begin{gathered} \forall i, j < m\ \forall \Gamma' \in [\Gamma]^{\mu}\ \exists \Gamma'' \in [\Gamma']^{\mu}\ \forall (F,G),(F',G') \in \Gamma \\ (\forall p < \omega) \neg (\eta_F(i) R_p \eta_{F'}(j)) \end{gathered}$$

where $\mu = (2^{\aleph_0})^+$. In fact, we can use the partition relation $\mu \to ((3)_\omega, \mu)^2$, noting that for each $p < \omega$ there do not exist (F,G), (F',G'), $(F'',G'') \in \Gamma$ such that $\eta_F(i) R_p \eta_{F'}(j)$, $\eta_F(i) R_p \eta_{F''}(j)$, $\eta_{F'}(i) R_p \eta_{F''}(j)$; otherwise we would get $x_{F'} = 0$. From (8) we see that we can assume

$$(9)\qquad \forall i, j < m\ \forall (F,G),(F',G') \in \Gamma\ \forall p < \omega\ (\eta_F(i) R_p \eta_{F'} j).$$

Hence by (4) we are finished again.

3.6. Claim. *If B_1, B satisfies the strong λ-chain condition, $a^x \in B$, B_2 is the result of a B-surgery of B_1 at a^x, then B_2 satisfies the strong λ-chain condition. If one of B_1, B satisfies the strong λ-chain condition, and the other only the λ-chain condition then B_2 satisfies the λ-chain condition.*

3.7. Claim. *If B_2 is the result of a B-surgery of B_1 at a^* then $B_1 < B_2$ (i.e. B_1 a subalgebra of B_2, and every maximal antichain of B_1 is a maximal antichain of B_2. This is also called "B_2 a regular extension of B_1").*

Proof. Trivial.

3.8. Claim. *The relation $<$ between Boolean algebra is a partial order and if B_i $(i < \alpha)$ is increasing continuous then $B_0 < \bigcup_{i<\alpha} B_i$, and if each satisfies the (strong) χ-chain condition, then so does $\bigcup_{i<\alpha} B$, for regular χ.*

Proof. See Solovay and Tenenbaum [19] for the χ-chain condition, and Kunen and Tall [5, p. 179] the strong χ-chain condition.

3.9. Claim. (1) *In the construction* 3.4, $\| B_i \| = \lambda$ *for* $i > 0$, $i < \lambda^+$.

(2) *In* 3.4, *if each $B_x(I)$ satisfies the strong χ-chain condition, χ regular, the $B = \operatorname{Sur}\langle I_i, a_i^* : i < \alpha \rangle$ satisfies the χ-chain condition.*

PROOF. (1) Trivial.

(2) By 3.5, 3.6, 3.7, 3.8.

3.10. LEMMA. (1) *The construction* 3.4, B_α, *is representable in* $M^*_{\aleph_0,\aleph_0}(\Sigma_{i<\alpha} I_i)$.

(2) *Moreover* $B_\alpha \restriction (1 - a^*_i)$ *is representable in* $M^*_{\aleph_0,\aleph_0}(\Sigma_{j<\alpha, j\neq i} I_j)$.

(3) *If* B_α *satisfies the* λ*-chain condition, then* B^c_α *(the completion of* B_α*) can be represented in* $M^*_{\lambda,\lambda}(\Sigma_{j<\alpha} I_j)$. *This representation can extend the one from* 3.10(1).

(4) *Similarly for* 3.10(2).

PROOF. (1) Define $f(0)=0$, $f(1)=1$. For $b \in B_\alpha$ and $b \neq 0,1$, say b first appears in B_{i+1}. Say

$$b = (b', \sum_{j<m} c_j d_j)$$

with $b' \in B_i \restriction (-a^*_i)$, $c_j \in B_i \restriction a^*_i$, $d_j \in B_{tr}(I_i)$. Say (by induction hypothesis) $f(b') = \tau$, $f(c_j) = \sigma_j$, $f(a^*_i) = \sigma$, $d_j = \rho_j(x_{\sigma'_1}, \cdots, x_{\sigma'_n})$. Then we set

$$f(b) = F_k(\sigma, \tau, \sigma_1, \cdots, \sigma_m, \sigma'_1, \cdots, \sigma'_n), \quad k \text{ codes} \langle m, n, \rho_0, \cdots, \rho_{m-1}\rangle$$

where F_k is a suitable function symbol. Thus, $f(b)$ codes all the relevant information about b.

(2) We may assume that $a^*_k \neq 0,1$. We go exactly as in (1), using $(-a^*_k)$ in place of 1, and working always with $B_\alpha \restriction (-a^*_k)$. Note that no terms involving I_k appear then.

(3) For each $a \in B^c_\alpha$ we can fix $\kappa < \lambda$ and a sequence $\langle b_\gamma : \gamma < \kappa \rangle$ of elements of B_α such that $a = \Sigma_{\gamma<\kappa} b_\gamma$; then let $f_a = F(\sigma_\gamma : \gamma < \kappa)$, where $f(b_\gamma) = \sigma_\gamma$ for all $\gamma < \kappa$.

(4) Similarly.

3.11. LEMMA. (1) *Suppose I is strongly* $(\aleph_0, \aleph_0, \psi_{tr})$*-unembeddable into J, B a Boolean algebra representable in* $M^*_{\aleph_0,\aleph_0}(J)$. *Then* $B_{tr}(I)$ *is not embeddable into B.*

(2) *Suppose I is strongly* (μ, κ, ψ_{tr})*-unembeddable into J and B a* B.A. *represented in* $M^*_{\mu,\kappa}(J)$. *Then* $B_{tr}(I)$ *is not embeddable into B.*

PROOF. (1) Let $G : B \to M^*_{\aleph_0,\aleph_0}(J)$ be a representation of B into $M^*_{\aleph_0,\aleph_0}(J)$, and h be an embedding of $B_{tr}(I)$ into B. For $\eta \in I$ define $f(\eta) = g(h(x_\eta))$. As I is strongly $(\aleph_0, \aleph_0, \psi_{tr})$-unembeddable into J, there are ν_1, ν_2, η, n such that $\nu_1 = \eta \restriction (n+1)$, $\nu_1 \restriction n = \nu_2 \restriction n$, $\nu_2(n) < \nu_1(n)$, $l(\nu_1) = l(\nu_2) = n+1$ and $\langle f(\nu_1), f(\eta)\rangle \approx \langle f(\nu_2), f(\eta)\rangle$ mod $M^*_{\aleph_0,\aleph_0}(J)$. Hence (because g is a representation) $h(x_\eta) < h(x_{\nu_1}) \Leftrightarrow h(x_\eta) < h(x_{\nu_2})$ (in B). But h is an embedding hence $x_\eta < x_{\nu_1} \Leftrightarrow x_\eta < x_{\nu_2}$ in $B_{tr}(I)$ contradicting the definition of $B_{tr}(I)$.

(2) Similar.

3.12. LEMMA. *Suppose I is strongly $(\lambda, \kappa, \psi_{\text{ptr}})$-unembeddable into J by f such that $f(\eta)$ is strongly finitary for $\eta \in I$, $l(\eta) = \omega$. If B is a Boolean algebra representable in $M^*_{\aleph_0,\aleph_0}(J)$ by g, $B \subseteq B_1$, B dense in B_1, g_1 extends g and is a representation of B_1 in $M^*_{\lambda,\kappa}(J)$, then $B_{\text{ptr}}(I)$ is not embeddable into B_1.*

PROOF. Suppose h is an embedding of $B_{\text{ptr}}(I)$ into B_1. For $\eta \in I$ define: if $l(\eta) < \omega$, $f(\eta) = g_1(h(x_\eta))$; if $l(\eta) = \omega$, choose $a_\eta \in B$, $0 < a_\eta < h(x_\eta)$ (possible as B is dense in B_1) and let $f(\eta) = g(a_\eta)$. As I is strongly $(\mu, \kappa, \psi_{\text{ptr}})$-unembeddable into J by f strong finitary on $\{\eta \in I : l(\eta) = \omega\}$ there are ν_1, ν_2, η, n such that $\nu_1 = \eta \restriction \eta^\wedge\langle\alpha\rangle$, $\nu_2 = \eta \restriction \eta^\wedge\langle\beta\rangle$, $\eta(n) = \langle\alpha, \beta\rangle$, $\alpha < \beta$ and

$$\langle f(\nu_1), f(\eta)\rangle \approx \langle f(\nu_2), f(\eta)\rangle \bmod M^*_{\mu,\kappa}(J).$$

Hence, as g_1 is a representation

$$(*) \qquad a_\eta < h(x_{\nu_1}) \Leftrightarrow a_\eta < h(x_{\nu_2}), \qquad a_\eta \cap h(x_{\nu_1}) = 0 \Leftrightarrow a_\eta \cap h(x_{\nu_2}) = 0.$$

But in B, $x_{\nu_1} \geqq x_\eta$, $x_{\nu_2} \cap x_\eta = 0$. Hence, as h is an embedding,

$$h(x_{\nu_1}) \geqq h(x_\eta), \qquad h(x_{\nu_2}) \cap h(x_\eta) = 0.$$

But $0 < a_\eta < h(x_\eta)$ so $h(x_{\nu_1}) > a_\eta$, $h(x_{\nu_2}) \cap a_\eta = 0$, contradiction to $(*)$ above.

We have proved that $B_{\text{ptr}}(I)$ is not embeddable into B.

3.13. CONCLUSION. Suppose K^ω_{tr} has the full strong $(\lambda, \lambda, \aleph_0, \aleph_0)$-$\psi_{\text{tr}}$-bigness property. Then:

(1) There is a rigid Boolean algebra B satisfying the $\aleph_1$-chain condition which has power λ.

(2) Moreover, if $a, b \in B$ are $\neq 0$, $a - b \neq 0$, then $B \restriction a$ cannot be embedded into $B \restriction b$ (hence B has no one-to-one endomorphism $\neq$ id).

(3) Moreover we can find such B_i $(i < 2^\lambda)$, $|B_i| = \lambda$; and if $a \in B_i$, $b \in B_j$, $i \neq j$ or $a - b \neq 0$ then $B_i \restriction a$ cannot be embedded into $B_j \restriction b$.

(4) Moreover, in (1), B has no one-to-one embedding $\neq$ id into B, and similarly for (2), (3).

PROOF. First note that if f is a one-to-one endomorphism $\neq$ id of any Boolean algebra B, then there is an element $a \neq 0$ with $B \restriction a$ isomorphic to $B \restriction f(a)$ and $a \cap f(a) = 0$. For, choose x with $x \neq f(x)$. If $x \cap -f(x) \neq 0$ we can take $a = x \cap -f(x)$; if $-x \cap f(x) \neq 0$ we can take $a = -x \cap f(x)$. Hence for (1) and (2) we only need to find B of power λ such that if $a, b \in B$ are non-zero and $a - b \neq 0$, then $B \restriction a$ cannot be embedded in $B \restriction b$.

Now let $\{I_\alpha : \alpha < \lambda\}$ exemplify the full strong $(\lambda, \lambda, \aleph_0, \aleph_0)$-$\psi_{\text{tr}}$-bigness property.

Let $B = \mathrm{Sur}\langle I_\alpha, a^*_\alpha : \alpha < \lambda\rangle$ be as in the construction 3.4, such that $B - \{0\} = \{a^*_\alpha : \alpha < \lambda\}$. Then by 3.9(1), $|B| = \lambda$. By 3.6, 3.8, 3.5, B satisfies the $\aleph_1$-chain condition. Now let $a, b \in B$ be non-zero, with $c = a - b \neq 0$. Suppose f is an embedding of $B \restriction a$ into $B \restriction b$. Then $f(c) \cap c = 0$, and $f \restriction (B \restriction c)$ is an embedding of $B \restriction c$ into $B \restriction f(c)$. But $c = a^*_\alpha$ for some α, hence $B_{\mathrm{tr}}(I_\alpha)$ is embeddable in $B \restriction c$, hence in $B \restriction f(c)$, hence in $B \restriction (-c) = B \restriction (-a_\alpha)$. But by 3.10(2), $B \restriction (-a^*_\alpha)$ is representable in $M^*_{\aleph_0,\aleph_0}(\Sigma_{\beta \neq \alpha, \beta < \lambda} I_\beta)$. This contradicts 3.11(1).

To prove (3), let A_i $(i < 2^\lambda)$ be subsets of λ, $|A_i| = \lambda$, and $A_i \not\subseteq A_j$ for $i \neq j$. For each $i < 2^\lambda$ let $\langle \gamma(i,\xi) : \xi < \lambda\rangle$ be an enumeration of A_i with each element repeated λ times. Let $\langle \alpha(i,\xi) : \xi < \lambda\rangle$ be one-to-one, $\alpha(i,\xi) < \lambda$, for each $i < 2^\lambda$. Let $\langle I_{\alpha,\gamma} : \alpha, \gamma < \lambda\rangle$ exemplify the full strong $(\lambda, \lambda, \aleph_0, \aleph_0)$-$\psi_{\mathrm{tr}}$-bigness property. Then let for $i < 2^\lambda$

$$B_i = \mathrm{Sur}\langle I_{\alpha(i,\xi),\gamma(i,\xi)}, a^*_{i,\xi} : \xi < \lambda\rangle,$$

where for each $\gamma \in A_i$, $B_i - \{0\} = \{a^*_{i,\xi}, \xi < \lambda, \gamma(i,\xi) = \gamma\}$. Now if $i < 2^\lambda$, $a, b \in B_i$, and $a - b \neq 0$, then $B_i \restriction a$ cannot be embedded in $B_i \restriction b$; this is proved as above. Now suppose $i \neq j$ both $< 2^\lambda$, $a \in B_i$, $b \in B_j$, and $0 \neq a, b$, f an embedding of $B_i \restriction a$ into $B_j \restriction b$. Choose $\gamma \in A_i - A_j$. Choose ξ so that $a^*_{i,\xi} = a$ and $\gamma(i,\xi) = \gamma$. Then $B_{\mathrm{tr}}(I_{\alpha(i,\xi),\gamma})$ is embeddable in $B_i \restriction a$, hence into $B_j \restriction b$. But by 3.11(2), $B_j \restriction b$ is representable in $M^*_{\aleph_0,\aleph_0}(\Sigma_{\eta<\lambda} I_{\alpha(j,\eta),\gamma(j,\eta)})$. Since $(\alpha(i,\xi), \gamma) \neq (\alpha(j,\eta), \gamma(j,\eta))$ for all $\eta < \lambda$, this contradicts 3.11(1).

3.14. THEOREM. *For λ uncountable and regular, there is a rigid* B.A. *B of power λ satisfying the $\aleph_1$-chain condition. Moreover, B has no one-one endomorphism $\neq$* id, *and if $a, b \in B$ are non-zero with $a - b \neq 0$, then $B \restriction a$ cannot be embedded in $B \restriction b$.*

PROOF. 2.1 and 3.1(3).

3.15. THEOREM. *For λ singular with $2^{\aleph_0} < \lambda$ and $\lambda^{\aleph_0} = \lambda$, the conclusion of* 3.14 *holds.*

PROOF. 2.6 and 3.13.

3.16. CONCLUSION. Suppose K^ω_{ptr} has the full strong $(\lambda, \lambda, (2^{\aleph_0})^+, (2^{\aleph_0})^+)$-$\psi_{\mathrm{ptr}}$-bigness property for f which are strongly finitary on P_ω. Then:

(1) There is a Boolean algebra B, $|B| = \lambda$ satisfying the $(2^{\aleph_0})^+$-chain condition with no one-to-one homomorphism from it to its completion (except the identity.)

(2) Moreover, for every disjoint non-zero a, b, there is no one-to-one homomorphism from $B \restriction a$ to $(B \restriction b)^c$ provided that $a - b \neq 0$.

(3) We can also get 2^λ such B.A.'s such that there is no one-to-one homomorphism from one to another (and even from $B_1 \restriction a$, $a \in B$, $a \neq 0$) to B_2 where $B_2 \neq B_1$ are in the family.

(4) If the full strong $(\lambda, \lambda, 2^{\aleph_0}, \aleph_1)$-$\psi_{\mathrm{ptr}}$-bigness property of $K^{\omega}_{\mathrm{ptr}}$ is exemplified by $\aleph_0$-stable I's, then we get Boolean algebras satisfying the countable chain condition and then we can replace $(\lambda, \lambda, (2^{\aleph_0})^+, (2^{\aleph_0})^+)$ by $(\lambda, \lambda, \aleph_1, \aleph_1)$.

REMARK. On how to get the $\aleph_1$-chain condition in more cases, see 6.2, 6.3.

PROOF. The proof is similar to 3.13.

We concentrate on (2), and w.l.o.g. $a \cap b = 0$. Let in 3.4 $x = \mathrm{ptr}$, and $\{I_\alpha : \alpha < \lambda\}$ exemplify the full strong $(\lambda, \lambda, (2^{\aleph_0}), (2^{\aleph_0})^+)$-$\psi_{\mathrm{ptr}}$-bigness property for f which are strongly finitary on P_ω.

Now, in the proof of 3.13, use ptr instead of tr, 3.12(2) instead of 3.11(1), and 3.5(2) instead of 3.5(1).

§4. On the narrowness of Boolean algebras

We prove in this section

4.1. THEOREM. (1) *If a Boolean algebra is λ-narrow (see below) then it has a dense subset of power $< \lambda$.*

(2) *If B does not have a dense subset of power $< \lambda$ then B has an irredundant set of λ pairwise incomparable elements.*

REMARK. (1) This completes a theorem of Baumgartner and Komjath which says the same for regular λ; of course we use their ideas.

(2) This answers a question from a preliminary version of [3].

4.2. DEFINITION. (1) A Boolean algebra B is λ-narrow if it has no λ pairwise incomparable elements, i.e., if $a_i \in B$ for $i < \lambda$ then for some $i \neq j < \lambda$, $a_i \leqq a_j$; we call a set of pairwise incompatible elements a pie.

(2) A set I of elements of a Boolean algebra B is irredundant if for every $b \in I$, $b \notin \langle I - \{b\}\rangle_B$ (i.e., b is not in the subalgebra of B which $I - \{b\}$ generates).

4.3. OBSERVATION. Suppose that $\{b_i : i < \alpha\} \subseteq B$ and for every $i < \alpha$

$$b_i \neq 0_B, \quad \neg(\exists x)[0 < x \in \langle b_j : j < i\rangle_B \wedge x \leqq b_i],$$

then for $i < j < \alpha$, $b_i \nleqq b_j$. Moreover $\{b_j : j < \alpha\}$ is irredundant.

PROOF OF 4.3. Well known (see [3] for references).

Clearly $b_i \not\leq b_j$ for $i < j < \alpha$. Now suppose $i < \alpha$ and $b_i \in \langle b_j : j < \alpha, j \neq i \rangle_B$. Then for some $n < \omega$, $j_0 < j_1 < \cdots < j_{n-1}$, $b_i \in \{b_{j_l} : l < n\}$, $i \notin \{j_l : l < n\}$, w.l.o.g. n is minimal, and obviously $n > 0$, $j_{n-1} > i$. Let B_0 be the subalgebra of B generated by $\{b_{j_l} : l < n-1\}$. Clearly B_0 is finite, hence atomic. As b_i is in the subalgebra generated by $\{B_0, b_{j_{n-1}}\}$, for every atom c of B_0,

$$b_i \cap c \notin B_0 \Rightarrow b_i \cap c \in \{c \cap b_{j_{n-1}}, c - b_{j_{n-1}}\}.$$

As $b_i \notin B_0$ (by the minimality of n) for some atom c of B_0, $b_i \cap c \notin B_0$ hence $c \cap b_i$, $c - b_i \neq 0$ and also $b_i \cap c \in \{c \cap b_{j_{n-1}}, c - b_{j_{n-1}}\}$. So $b_{j_{n-1}} \cap c \notin B_0$ and belongs to $\{c \cap b_i, c - b_i\}$. Now both $\{c \cap b_i, c - b_i\}$ are not zero, both are in $\langle B_0 \cup \{b_i\} \rangle \subseteq \langle \{b_\alpha : \alpha < j_{n-1}\} \rangle$ and one of them is $\leq b_{j_{n-1}}$, contradiction to the choice of $b_{j_{n-1}}$.

Proof of Theorem 4.1. Suppose λ is minimum such that 4.1(2) fails, with B a counterexample. Then

(1) $\forall 0 \neq a \in B \, \exists 0 \neq b \leq a$ ($B \restriction b$ has a dense subset of power $< \lambda$).

For suppose not, and let a be a counterexample. Clearly a is not an atom, so choose disjoint non-zero b_1, $b_2 \leq a$. We can define inductively on $i < \lambda$ (for $l = 1,2$) $c_i^l \leq b_l$ such that

(*) $$\neg \exists x (0 < x \in \langle c_j^l : j < i \rangle_{B \restriction b_l} \wedge x \leq c_i^l)$$

(If we cannot define c_i^l this means that $\langle c_j^l : j < i \rangle_{B \restriction b_l}$ is a dense subset of $B \restriction b_l$, contradicting the choice of a.) Hence by (*) and 4.3 we have $c_i^l \not\leq c_j^l$ for $i < j < \lambda$, and $\{c_j^l : j < \lambda\}$ is irredundant. Hence $\{c_i^1 \cup (b_2 - c_i^2) : i < j\}$ is an irredundant pie in B of size λ, contradicting the choice of λ. Thus (1) holds.

Let X be an infinite maximal set of pairwise disjoint elements of B such that $\forall b \in X$ ($B \restriction b$ has a dense subset of power $< \lambda$). By (1), X is a maximal set of pairwise disjoint elements of B. Now X is an irredundant pie, so $|X| < \lambda$ by the choice of λ. Furthermore, since B has no dense subset of power $< \lambda$, clearly $\mathrm{cf}\,\lambda \leq |X|$. In particular, λ is singular. Say $\lambda = \sum_{\alpha < \mathrm{cf}\lambda} \mu_\alpha$ with each $\mu_\alpha < \lambda$. Now since B has no dense subset of power $< \lambda$, clearly $\forall \alpha < \mathrm{cf}\,\lambda$ $\forall Y \in [X]^{<\mathrm{cf}\lambda}$ $\exists b \in X - Y$ ($B \restriction b$ has no dense subset of power $< \mu_\alpha$). Hence we can construct by induction $\langle b_\alpha : \alpha < \mathrm{cf}\,\lambda \rangle$ by choosing $b_\alpha \in X - \{b_\beta : \beta < \alpha\}$ so that $B \restriction b_\alpha$ has an irredundant pie D_α of power μ_α, using the minimality of λ. But then $\bigcup_{\alpha < \mathrm{cf}\lambda} D_\alpha$ is an irredundant pie in B of power λ, contradiction.

4.4. Claim. *If* $\mathrm{cf}\,\lambda > \aleph_0$ *then by* ccc *forcing* P, *we can introduce a* B.A. *of power* λ *which has a pie* A_μ *of power* μ *for every* $\mu < \lambda$ *but has no pie of power* λ ($2^{\aleph_0}$ *will be* $> \lambda$).

REMARK. Similar forcing but κ-complete does the job above $2^{\aleph_0}$.

PROOF. Let $\lambda_\alpha < \lambda$ for $\alpha < \operatorname{cf}\lambda = \kappa$, $\lambda = \Sigma_{\alpha<\kappa}\lambda_\alpha$. We shall concentrate on the case λ is limit, so we can assume $\lambda_\alpha > \Sigma_{i<\alpha}\lambda_i$, λ_α regular. Let $I_\alpha = \{i : \bigcup_{\beta>\alpha}\lambda_\beta \leqq i < \lambda_\alpha\}$, so I_α $(\alpha < \kappa)$ is a partition of κ.

Let us define the partial order P: an element is a pair (B, w), B a finite B.A. generated by a finite set $\{x_i : i \in w\}$, $w \subseteq \lambda$, such that:

(1) in B, for each $i \in w$, x_i is not in the subalgebra generated by $\{x_j : j \in w, j \neq i\}$;

(2) if i, j are in the same interval, then x_i, x_j are incomparable.

FACT A. P satisfies the λ-ccc.

PROOF. As in [15] §2: w.l.o.g. we have $w = w_1 \cap w_2$, $(B, w) < (B_l, w_l)$; and there is an isomorphism f from B_1 onto B_2 such that:

(α) $\{f(x_i) : i \in w_1\} = \{x_j : j \in w_2\}$ and $i \in w_1 \Rightarrow f(x_i) = x_i$.

(β) Let $u_l = \{\alpha : (\exists i \in w_l)\ i \in I_\alpha\}$. If $i \in I_\alpha$, $\alpha \in u_1 \cap u_2$ then $f(x_i) \in \{x_j : j \in I_\alpha\}$.

It is enough to prove they are compatible.

Let B^* be the free product of B_1, B_2 over B, $w^* = w_1 \cup w_2$; then easily (B^*, w^*) is the required upper bound. (*Proof*: As there, check inside every atom of B.)

FACT B. For every $i < \lambda$, $D_i = \{(B, w) : i \in w\} \subset P$ is dense.

FACT C. If $G \subset P$ is directed, $G \cap D_i \neq \emptyset$ for every $i < \lambda$, $B^G = \bigcup\{B : (B, w) \in G\}$, *then* B^* is a B.A. generated by $\{x_i : i < \lambda\}$ and $A_\alpha = \{x_i : i \in I_\alpha\}$ is a pie of power λ_α.

FACT D. If $G \subseteq P$ is generic, then $B^G \in V[G]$ has no pie of power λ.

PROOF. Suppose $\langle \boldsymbol{a}_i : < \lambda\rangle$ is a P-name of such pie (a list with no repetitions). The case, λ is regular, is just like the proof in [15] that the generic B.A. has no $\aleph_1$-pie. So let λ be singular.

For each $\alpha < \kappa$, $i \in I_\alpha$ there is $p_i = (B_i, w_i)$ s.t. $p_i \Vdash \boldsymbol{a}_i = a_i$ for some $a_i \in B_i$. For some $J_\alpha \subseteq I_\alpha$, $|J_\alpha| = \lambda_\alpha$, $\{w_i : i \in J_\alpha\}$ is a Δ-system, $\bigcap_{i\in J_\alpha} w_i = w^\alpha$ and for some B^α, $(B^\alpha, w^\alpha) \subseteq (B_i, w_i)$ for every $i \in J_\alpha$ and $w_i \cap w_j = w^\alpha$ for $i \neq j \in J_\alpha$. Also we can assume $w_i \cap (\bigcup_{\beta<\alpha}\lambda_\beta) \subseteq w^\alpha$, and for $i, j \in J_\alpha$, B_i, B_j are isomorphic over B^α with an isomorphism taking a_i to a_j satisfying (α), (β) from Fact A. Now κ is regular $> \aleph_0$, so we find an unbounded $S \subset \kappa$, and (B, w), such that the following holds: $\alpha \in S \Rightarrow (B, w) \subseteq (B^\alpha, w^\alpha)$ and $\alpha < \beta$, $\alpha \in S$, $\beta \in S$ implies the following: $w^\alpha \cap w^\beta = w$ and $w^\alpha \subseteq \lambda_\beta$ and B^α, B^β are isomorphic over B, the isomorphism satisfies (α), (β) and this isomorphism can be extended to an isomorphism of B_i, B_j for any $i \in J_\alpha$, $j \in J_\beta$ s.t. a_i correspond to a_j.

Now if we choose $\alpha < \beta$ in S, $i \in J_\alpha$, $j \in J_\beta$ and try to amalgamate (B_i, w_i), (B_j, w_j) such that a_i, a_j are comparable, we succeed. Note that (2) of the definition of P has no effect, so we can proceed as in [15].

§5. On endo-rigid indecomposable Bonnet-rigid B.A's

5.1. Definition. (1) For a B.A. B, we call $\psi = \psi(x, c, \bar{b})$ a p.s. (possible support) over B if $\bar{b} = \langle b_0, b_1, \cdots, b_n \rangle$ is a partition of 1 (by elements of B), $c \leqq b_0$, $b_1 \neq 0, \cdots, b_n \neq 0$, and ψ is the formula

$$(x \cap b_0 = c) \wedge \bigwedge_{l=1}^{n} (0 < x \cap b_l < b_l).$$

The p.s. is *degenerate* if $n = 0$. For $n = 1$ we write b_0 only rather than $\langle b_0, b_1 \rangle$.

(2) For p.s. $\psi^l = \psi^l(x, c^l, \bar{b}^l)$, $l = 1, 2$, we write $\psi^1 < \psi^2$ (ψ^2 extends ψ^1) if $c^1 \leqq c^2$, $b_0^1 - c^1 \leqq b_0^2 - c^2$, and for every i, $1 \leqq i \leqq n^1$, there is a j, $1 \leqq j \leqq n^2$, such that $b_i^2 \leqq b_j^1$ (where $\bar{b}^l = \langle b_i^l : i < n^l \rangle$, of course) (the first two conditions are equivalent to $b_0^1 \leqq b_0^2$ and $c^1 = b_0^1 \cap c^2$).

(3) For a p.s. ψ and extensions $\psi^1, \cdots, \psi^n$ of it, we say that $\langle \psi^1, \cdots, \psi^n \rangle$ is a *disjoint* system of extensions of ψ if $b_0 = b_0^i \cap b_0^j$ for $i \neq j$, and there do not exist i and $l \geqq 1$ such that $b_l^i \leqq \bigcup_{j=1}^{n} b_0^j$.

5.2. Definition. For a 1-type p (always quantifier free in the language of B.A.'s for this section) and a B.A. B, we say that *B absolutely omits p* if for every p.s. $\psi = \psi(x, c, \bar{b})$ and every $\beta < \omega$ there is a disjoint system of extensions ψ^i ($i < \beta$) of ψ and formulas $\vartheta^i(x)$ from p such that $\psi^i \vdash \neg \vartheta^i(x)$ for all $i < \beta$.

5.3. Claim. *If B absolutely omits p, then B omits p.*

Proof. Suppose c realizes p. Let $\psi = \psi(x, c, 1)$; ψ is a degenerate p.s. Hence with $\beta = 1$ we get ψ^0 extending ψ and $\vartheta^0(x) \in p$ such that $\psi^0 \vdash \neg \vartheta^0(x)$. But $\psi = \psi^0$ is the formula $x = c$, contradiction.

5.4. Remark. (1) Definition 5.2 is analogous to Rubin [8], in which $\beta = 1$. In fact this case ($\beta = 1$) is the specification of Keisler's omitting type theorem for $L(Q)$ to atomless Boolean algebras with the added axiom $(\forall x)$ $[x > 0 \rightarrow Qy(y < x)]$.

(2) We can do similar work for n-types, but no need arises.

5.5. Claim. *For atomless B, B absolutely omits p iff for every pair $c \leqq b_0$ and* *every $\beta < \omega$ there exist c^i, b^i $(i < \beta)$ such that $b^i \cap b^j = b_0$ for $i \neq j$, $b_0 \leqq b^i$, $c \leqq c^i$, $b_0 - c \leqq b^i - c^i$, and $(\forall i < \beta)[x \cap b^i = c \vdash \neg \vartheta^i(x)]$ for some $\vartheta^i(x) \in p$.*

PROOF. $\Rightarrow$ Given a pair $c \leqq b_0$ and $\beta < \omega$, let $\psi = \psi(x, c, b_0)$ and by 5.2 let ψ^i $(i < \beta)$ be a disjoint system of extensions of ψ and $\vartheta^i(x)$ $(i < \beta)$ a system of formulas from p such that $\psi^i \models \neg \vartheta^i(x)$ for all $i < \beta$. Say $\psi^i = \psi^i(x, c^i, \bar{b}^i)$. Now we use the following fact from the theory of B.A.'s:

(*) If A is an atomless B.A. and $\langle d_0, \cdots, d_{m-1}\rangle$ $(m \in \omega)$ a system of non-zero elements of A, then there exist pairwise disjoint non-zero $e_0, \cdots, e_{m-1}$ such that $e_i \leqq d_i$ for all $i < m$.

We hence get elements e^i_l $(i < \beta, 1 \leqq l \leqq n_i)$ which are non-zero and pairwise disjoint, with $e^i_l \leqq b^i_l - \bigcup_{j<\beta} b^j_0$ $(i < \beta, 1 \leqq l \leqq n_i)$. Now choose disjoint non-zero $t_{i,l,0}$, $t_{i,l,1} \leqq e^i_l$. Then the system $\langle\langle c^i \cup \bigcup_{l=1}^{n_i} t_{i,l,0}, b^i_0 \cup \bigcup_{l=1}^{n_i} (t_{i,l,0} \cup t_{i,l,1})\rangle : i < \beta\rangle$ is as desired — the checking is easy.

$\Leftarrow$. Let $\psi = \psi(x, c, \langle b_0, \cdots, b_n\rangle)$ be a p.s. and let $\beta < \omega$. Choose c^i, b^i for all $i < \beta + n$ so that $b^i \cap b^j = b_0$ for $i \neq j$, $c \leqq c^i$, $b_0 - c \leqq b^i - c^i$ and for all $i < \beta + n$, $x \cap b^i = c^i \vdash \neg \vartheta^i$ for some $\vartheta^i \in p$. By rearranging the c^i, b^i, we may assume that if $1 \leqq l \leqq n$ and $b^i \cap b_l \neq 0$ for some $i < \beta + n$ then $b^i \cap b_l \neq 0$ for some $i \geqq \beta$. After this rearrangement, $\langle \psi(x, c^i, \langle b^i, b_1 - b^i, \cdots, b_n - b^i\rangle) : i < \beta\rangle$ is as desired (easily checked.)

Now let B be a countable atomless B.A., I a maximal ideal of B, and $D = B - I$. Let $P_I = P_D = \{x \cap b = c : c \leqq b \in I\}$. We partially order P_I by setting $(x \cap b = c) \leqq (x \cap b' = c')$ iff $b \leqq b'$, $c \leqq c'$ and $b \cap c' = c$. A subset G of P_D is an *ideal* if it is directed upwards and

$$(x \cap b = c) \leqq (x \cap b' = c') \in G \Rightarrow (x \cap b = c) \in G.$$

Let $\mathrm{Gen}(P_D) = \{G : G \subseteq P_D,\ G \text{ an ideal}\}$. A natural topology on $\mathrm{Gen}(P_D)$ is defined by the basic open sets $\{G : (x \cap b = c) \in G\}$, for $c \leqq b \in I$. This topology is not Hausdorff, but it satisfies the Baire category theorem: a countable intersection of open dense sets is dense. For almost all G means for all G in a countable intersection of open dense sets. We use P for P_I.

Given an ideal G in P_D, $B[G]$ is the B.A. freely extending B by a new element t, subject to $t \cap b = c$ for $(x \cap b = c) \in G$.

5.6. MAIN LEMMA. *Let B, I, D be as above, and also suppose that p is a 1-type absolutely omitted by B.*

(1) *For all $b \in I$, for almost all G there is a c such that $(x \cap b = c) \in G$.*

(2) *For all $b \notin I$, for almost all G there is a $b' \leqq b$ and $c' < b'$ such that $b' \in I$, $c' \neq 0$, and $(x \cap b' = c') \in G$; similarly with $c' = 0$.*

(3) *For almost all G, B is dense in $B[G]$; $B[G]$ is atomless, and $t \notin B$.*

(4) *For almost all G, $B[G]$ absolutely omits p.*

REMARK. We shall frequently use properties which hold for almost all G.

NOTATION. Let $(a, x)^t$ stand for $a \cap x$ if $t = 0$ and $a - x$ if $t = 1$.

PROOF OF 5.6. For $c \leqq b \in I$ let $U_{b,c}$ denote the basic open set $\{G: (x \cap b = c) \in G\}$.

(1) Given $b \in I$, the set $V = \{G: \exists c \leqq b(x \cap b = c) \in G\}$ is clearly open. It is dense, since if $c' \leqq b' \in I$, then $(x \cap b' = c') \leqq (x \cap (b \cup b') = c')$ and $(x \cap b = b \cap c') \leqq (x \cap (b \cup b') = c')$, and so if $(x \cap (b \cup b') = c') \in G$ we have $G \in U_{b',c'} \cap V$.

(2) Given $b \notin I$, the set $V = \{G: \exists b' \leqq b\, \exists c' \leqq c(b' \in I$ and $c' \neq b', 0$ and $(x \cap b' = c') \in G)\}$ is clearly open. It is dense, since if $c_0 \leqq b_0 \in I$, then $b \nleqq b_0$, and hence, since B is atomless, there is a $b' \leqq b \cap -b_0$ with $0 \neq b' \in I$. Choose $c' \in B$, $0 < c' < b'$. Then

$$(x \cap b' = c') \leqq (x \cap (b_0 \cup b') = c_0 \cup c') \quad \text{and}$$

$$(x \cap b_0 = c_0) \leqq (x \cap (b_0 \cup b') = c_0 \cup c'),$$

and so if $(x \cap (b_0 \cup b') = c_0 \cup c') \in G$ then $G \in U_{b_0,c_0} \cap V$. For $c' = 0$ we use $x \cap (b_0 \cup b') = c_0$.

(3) Take G satisfying (1) and (2) (both clauses) for all $b \in I$ and all $b \notin I$. Without loss of generality we may take the non-zero elements of $B[G]$ in four forms: $t \cap b$ with $b \in I$, treated by (1); $t \cap b$ with $b \notin I$, treated by (2), first clause; $-t \cap b = b - (t \cap b)$ with $b \in I$, treated by (1); $-t \cap b$ with $b \notin I$ treated by (2), second clauses: $-t \cap b \geqq -t \cap b' = b' - (t \cap b') = b'$. So we have proved "$B$ is dense in $B[G]$". It follows that $B[G]$ is atomless.

To show that for almost all G, $t \notin B$, let $d \in B$; it suffices to show that for almost all G, $t \neq d$. Let $V = \{G: \exists c\, \exists b(c \leqq b \in I)$ and $c \neq b \cap d$ and $(x \cap b = c) \in G\}$. Thus V is open, and clearly $t \neq d$ for any $G \in V$. To show that V is dense, let $c' \leqq b' \in I$. Choose b'' with $b' < b'' \in I$. Choose c^* with $c^* \leqq b'' - b'$, $c^* \neq (b'' - b') \cap d$. Then

$$(x \cap b' = c') \leqq (x \cap b'' = c' \cup c^*),$$

and if $(x \cap b'' = c' \cup c^*) \in G$ then $G \in V$.

(4) First we claim:

5.6A. FACT. For almost all G, if $c \leqq b_0$ in $B[G]$ then there are disjoint d_0, e_0, and $\zeta \in \{0,1\}$ and disjoint d, e and $\xi \in \{0,1\}$ such that $b_0 = d_0 \cup (e_0, t)^\zeta$, $c = d \cup (e,t)^\xi$, and one of the following holds:

(a) $e = e_0 \in D$ and $\zeta = \xi$, $d \leqq d_0$,

(b) $e = 0$, $d \leqq d_0$, and $e_0 \in D$,

(c) $e_0 = 0$, d, $e \leqq d_0$, and $e \in D$,

(d) c, $b_0 \in B$.

For, let G satisfy (1) for *all* $b \in I$. For any $d \in B[G]$ write

$$d = e_0 \cup (e_1 \cap t) \cup (e_2 - t) \qquad (e_0, e_1, e_2 \in B)$$

with e_0, e_1, e_2 pairwise disjoint. If $e_2 \in I$, then $e_1 - t = e_2 - (e_2 \cap t) \in B$; if $e_2 \notin I$, then $e_1 \cap t = e_1 \cap (-e_2 \cap t) \in B$. Thus each $d \in B[G]$ can be written in the form $e_0 \cup (e_1, t)^\tau$. So, write $b_0 = d_0 \cup (e_0, t)^\zeta$ with $d_0 \cap e_0 = 0$ and $\zeta \in \{0,1\}$, and $c = d \cup (e,t)^\xi$ with $d \cap e = 0$ and $\xi \in \{0,1\}$. Since $f \cap t \in B$ and $f - t = f - (f \cap t) \in B$, when $f \in I$, we may assume that $e \in D$ if $e \neq 0$, and $e_0 \in D$ if $e_0 \neq 0$. If $e = e_0 = 0$ we are in case (d). Assume $e_0 = 0$, $e \neq 0$. Then, as $(c \leqq b_0)$ $(e,t)^\xi \leqq d_0$ so $c = d \cup (e \cap d_0, t)^\xi$ and we are in case (d) if $e \cap d_0 \in I$, otherwise in case (c). Assume $e = 0$, $e_0 \neq 0$. Then $b_0 = d_0 \cup (e_0 \cap d) \cup (e_0 - d, t)^\xi$, giving (d) or (b). Finally, suppose e, $e_0 \neq 0$. If $\zeta = 0$ and $\xi = 1$, then w.l.o.g. $e = e_0$ hence $c \cap e$, $b_0 \cap e$ are disjoint and non-zero, contradicting $c \leqq b_0$; similarly if $\zeta = 1$ and $\xi = 0$. So, assume $\zeta = \xi$, so we can get (a). Thus 5.6A holds.

Now to prove 5.6(4) we apply 5.5. Let G be such that (1), (2), (3), 5.6A hold. Suppose $c \leqq b_0$ in $B[G]$ and $\beta < \omega$. Write c and b_0 as in 5.6A.

Case (a). $e = e_0 \in D$ and $\zeta = \xi$. Let $(x \cap b^* = c^*) \in P$; and w.l.o.g. $e = -b^*$; we shall find a bigger element of P such that for every G to which it belongs there exist c^i, b^i $(i \leqq \beta)$ as desired in 5.5. Note that $d \cup (e_0 \cap b^*, c^*)^\zeta \leqq d_0 \cup (e_0 \cap b^*, c^*)^\zeta \in I$. Applying 5.5 to this pair in B, we get c^i, b_0^i $(i \leqq \beta)$ such that $b_0^i \cap b_0^j = d_0 \cup (e_0 \cap b^*, c^*)^\zeta$ for $i \neq j$, $d \cup (e_0 \cap b^*, c^*)^\zeta \leqq c^i \leqq b_0^i$, $d_0 - d \leqq b_0^i - c^i$, and $\forall i \leqq \beta$, $x \cap b_0^i = c^i \vdash \neg \vartheta^i(x)$ for some $\vartheta^i(x) \in p$. Now

$$b_0^i \cap b_0^j = d_0 \cup (e_0 \cap b^*, c^*)^\zeta \in I \qquad \text{for } i \neq j,$$

so there is at most one $i \leqq \beta$ such that $b_0^i \notin I$. So w.l.o.g. $\forall i < \beta$ $(b_0^i \in I)$. Let $b^{**} = b^* \cup \bigcup_{i<\beta} b_0^i$, and if $\zeta = 0$ let $c^{**} = c^* \cup \bigcup_{i<\beta}(c^i - b^*)$, while if $\zeta = 1$ let $c^{**} = c^* \cup \bigcup_{i<\beta}(b_0^i - c^i - b^*)$. Thus $(x \cap b^* = c^*) \leqq (x \cap b^{**} = c^{**}) \in P$. Now suppose $(x \cap b^{**} = c^*) \in G$. Set

$$b_1^i = b_0^i \cup (e,t)^\zeta, \quad c_1^i = c^i \cup (e,t)^\zeta \qquad \text{for } i < \beta;$$

we claim that the demand in 5.5 holds for these elements in $B[G]$. For $i \neq j$ we have

$$b_1^i \cap b_1^j = (b_0^i \cap b_0^j) \cup (e,t)^\zeta = d_0 \cup (e \cap b^*, c^*)^\zeta \cup (e,t)^\zeta$$

$$= d_0 \cup (e,t)^\zeta = b_0$$

since $t \cap b^* = c^*$. Clearly $c \leqq c_1^i \leqq b_1^i$ and $b_0 - c \leqq b_1^i - c_1^i$. Finally, suppose that $x \cap b_1^i = c_1^i$. Then

$$x \cap b_0^i \cap -(e,t)^\zeta = x \cap b_1^i \cap -(e,t)^\zeta \qquad \text{by } b_1^i\text{'s definition}$$

$$= c_1^i \cap -(e,t)^\zeta \qquad \text{by the hypothesis on } x$$

$$= c^i \cap -(e,t)^\zeta \qquad \text{by } c_1^i\text{'s definition.}$$

Also $(e,t)^\zeta \leqq c_1^i \leqq x$. So to show $x \cap b_0^i = c^i$ and hence finish case (a) it suffices to show $b_0^i \cap (e,t)^\zeta = c^i \cap (e,t)^\zeta$. Assume $\zeta = 0$. Now if $i \neq j$, then

$$b_0^i \cap e \cap t \cap (c^j - b^*) = b_0^i \cap b_0^j \cap e \cap t \cap (c^j - b^*) \quad \text{as } b_0^j \cap c^j = b_0^j \cap c^j$$

$$= (d_0 \cup (e \cap c^*)) \cap e \cap t \cap (c^j - b^*) = 0$$

since $d_0 \leqq b^*$ and $t \cap b^* = c^*$. Hence

$$b_0^i \cap e \cap t = b_0^i \cap e \cap t \cap b^{**} \quad \text{as } t \cap b^{**} = c^{**}$$

$$= b_0^i \cap e \cap t \cap c^{**}$$

$$= b_0^i \cap e \cap t \cap (c^* \cup (c^i - b^*))$$

by the previous computation

$$\leqq e \cap t \cap c^i \quad \text{since } e \cap c^* \leqq c^i$$

$$\leqq b_0^i \cap e \cap t, \quad \text{as } c^i \cap e = b_0^i \cap e \quad \text{as } e = -b^*$$

as desired. On the other hand, if $\zeta = 1$ then

$$(b_0^j - c^j - b^*) \cap (e - t) \leqq c^{**} \cap (e - t) \leqq b^{**} \cap t \cap (e - t) = 0,$$

and now

$$b_0^i \cap (e - t)$$

$$= b_0^i \cap (e - t) \cap b^{**} \quad \text{as } b_0^i \leqq b^{**}$$

$$= b_0^i \cap (e - t) \cap (b^{**} - c^{**}) \quad \text{as } t \cap b^{**} = c^{**}$$

$$= b_0^i \cap (e - t) \cap -c^* \cap \bigcap_{j<\beta} (-b_0^j \cup c^j \cup b^*) \text{ by the previous computation}$$

$$= b_0^i \cap (e - t) \cap (-t \cup -b^*) \cap (b^* \cup \bigcap_{j<\beta} (-b_0^j \cup c^j)) \quad \text{as } t \cap b^* = c^*$$

$$\leqq (b_0^i \cup (e - t) \cap b^*) \cup (b_0^i \cap (e - t) \cap \bigcap_{j<\beta} (-b_0^j \cup c^j))$$

$$\leqq (b_0^i \cap (e - t) \cap (b^* - c^*)) \cup (b_0^i \cap (e - t) \cap (-b_0^i \cup c^i))$$

$$\leqq (e - t) \cap c^i \leqq b_0^i \cap (e - t) \quad \text{as desired.}$$

Case (b). $e = 0$, $d \leqq d_0$, and $e_0 \in D$. We find c^i, b_0^i $b_0 (i \leqq \beta)$ as in Case (a), for the pair $d \leqq d_0 \cup (e \cap b^*, c^*)^\zeta$ and define b^{**} as there. This time let $c^{**} = c^*$, if $\zeta = 0$, and $c^{**} = c^* \cup \bigcup_{i<\beta} (b_0^i - b^*)$ if $\zeta = 1$. Then

$$(x \cap b^* = c^*) \leqq (x \cap b^{**} = c^{**}) \in P.$$

Suppose $(x \cap b^{**} = c^{**}) \in G$. Set $b_1^i = b_0^i \cup (e, t)^\zeta$; we claim that c^i, b_1^i $(i < \beta)$ are as desired in 5.5 for $B[G]$. This is proved much as in Case (a).

Case (c). $e_0 = 0$, $e \leqq d_0$, and $e \in D$. We assume $e = b^*$, and use $d = c \leqq d_0 - e$ to find c^i, $b_0^i \in I$ as in Case (a). Then define b^{**}, c^{**} exactly as in Case (a). This time we use $b_1^i = b_0^i \cup e$ and $c_1^i = c^i \cup (e, t)^\zeta$. The details are similar to those in Case (a).

Case (d). c, $c_0 \in B$. This is trivial, since B strongly omits p.

5.7. CLAIM. *Let B be a countable atomless* B.A., *$f : B \to B$ an endomorphism, Γ a countable set of 1-types which B absolutely omits, $I = \{x : x = y \cup z, f(y) = 0, \forall v \leqq z (f(v) = v)\}$. Thus I is an ideal. Assume that B / I is infinite.*

Then there is $B^* \supseteq B$, $t \in B^* - B$, such that B^* is countable atomless and

(1) B^* absolutely omits every $q \in \Gamma$ (B dense in B^*),

(2) B^* absolutely omits $p = \{x \cap f(a) = f(c);\ a,\ c \in B,\ t \cap a = c\}$.

(Thus in no $B^{**} \supseteq B^*$ which omits p can f be extended to an endomorphism; otherwise $f(t)$ will realize p.)

PROOF. There is a maximal ideal J of B which contains every a such that a / I is zero or a finite union of atoms. Let D be the dual filter. Note that if $a \in B - J$ then there are b, c such that $a = b \cup c$, $b \cap c = 0$, $b \in B - J$, and $c \in B - I$. We can repeat this process on b, noting that $c_1 \cap c_2 = 0 \Rightarrow f(c_1) \cap f(c_2) = 0$. This means that we may assume that $f(c) \in J$.

Applying Lemma 5.6 to each $q \in \Gamma$, we see that for almost all G, (1) holds. Now we show that almost every G satisfies (2). Let G satisfy (1), (2), (3), (4) in 5.6 and its proof. Suppose $c \leqq b_0$ in $B[G]$ and $\beta < \omega$. Write c and b_0 as in Fact

5.6A. Suppose $(x \cap b^* = c^*) \in P$; we want to find $x \cap b^{**} = c^{**}$ as in the proof of 5.6, suitable for p of (2). Let $B^0 = \langle d, d_0, e, e_0, c^*, b^* \rangle_B$. We may assume that $u \leqq b^*$ whenever $u \in B^0 \cap J$ and $e = -b^*$ or $e = 0$; similarly for e_0. Now using the remark in the first paragraph we can define by induction on $n < \omega$ elements $a_0, \cdots, a_n, \cdots \leqq -b^*$ such that $a_l \in J - I$, $f(a_l) \in J$, and a_n disjoint from $b^* \cup a_0 \cup f(a_0) \cup \cdots \cup a_{n-1} \cup f(a_{n-1})$. As in [14] (or see Monk [7], lemma 8), we can assume $a_n \not\geqq f(a_n)$ (replacing a_n by some $a'_n \leqq a_n$). Now by Ramsey's theorem we can assume

(1) the truth values of $f(a_n) \cap -a_n \cap -b^* \cap a_m = 0$, $f(a_n) \cap -a_n \cap x = 0$, $f(a_n) \cap x = 0$ (for each $x \in B^0$) depend only on whether $n < m$, $n = m$, or $n > m$.

(2) either (A) $\forall n (f(a_n) \cap -a_n \leqq b^*)$ or (B) $\forall n (f(a_n) \cap -a_n \cap -b^* \neq 0)$.

By (1) and possibly replacing $\langle a_n : n < \omega \rangle$ by $\langle a_{2n+2} : n < \omega \rangle$ we get

(3) if $f(a_n) - a_n - x \neq 0$, $x \in B^0$, $k < \omega$ then $f(a_n) - a_n - x - \bigcup_{m<k} a_m \neq 0$.

We consider now several cases, defining b^{**}, c^{**}, b_0^i, c^i $(i < \beta)$ in each case so that if $(x \cap b^{**} = c^{**}) \in G$, then $\langle b_0^i, c^i \rangle$ exemplify 5.5's criterion in $B[G]$.

For proving the existence of a suitable ϑ^i, we let $i < \beta$, $B[G] \subseteq B^1$, $x \in B^1$, $x \cap b_0^i = c^i$ and assume x realizes p and we shall get a contradiction.

For notational simplicity assume $\zeta = 0$. Let

(i) $b^{**} = b^* \cup \bigcup_{l<2\beta} (a_l \cup f(a_l))$,

(ii) $c^{**} = c^* \cup \bigcup_{l<\beta} a_{2l}$,

and in $B[G]$ let:

(iii) $b_0^i = b_0 \cup f(a_{2i}) \cup f(a_{2i+1})$,

(iv) $c^i = c$.

Case A. $f(a_j) - a_j \leqq d$ (for some j)

So this holds for every j.

Now $t \cap a_{2i+1} = 0$ hence (as x realizes P) $x \cap f(a_{2i+1}) = 0$, so

$$0 = x \cap f(a_{2i+1}) = x \cap (f(a_{2i+1}) \cap b_0^i) = (x \cap b_0^i) \cap f(a_{2i+1})$$
$$= c^i \cap f(a_{2i+1}) = c \cap f(a_{2i+1}) \geqq c \cap (f(a_{2i+1}) - a_{2i+1}).$$

So $c \cap (f(a_{2i+1}) - a_{2i+1}) = 0$, and $c \geqq d$ and $d \geqq f(a_{2i+1}) - f(a_{2i+1})$ (a hypothesis of the case A). Hence $f(a_{2i+1}) - a_{2i+1} = 0$. A contradiction to the choice of the a_n's.

Case B. For no j, $f(a_j) - a_j \leqq d$

Clearly $a_{2i} = t \cap a_{2i}$ hence

$$f(a_{2i}) = x \cap f(a_{2i}) = x \cap (f(a_{2i}) \cap b_0^i) = (x \cap b_0^i) \cap f(a_{2i})$$
$$= c^i \cap f(a_{2i}) = c \cap f(a_{2i}),$$

so $f(a_{2i}) \leqq c$. But by the hypothesis of the case $f(a_{2i}) - d \neq 0$. Remember that $c = d \cup (e, t)^\zeta$, so necessarily $e \neq 0$ hence $e = -b^*$, and by the choice of c^{**} (which influence c through t) $f(a_{2i}) - b^* - \bigcup_{m<\beta} a_{2m} = 0$. By (3) this implies $f(a_{2i}) - a_{2i} \leqq b^*$, and as $f(a_{2i}) \leqq c$ clearly $f(a_{2i}) - a_{2i} \leqq b^* \cap c$. But as $c = d \cup (e, t)^\zeta$, $e = -b^*$, this implies $f(a_{2i}) - a_{2i} \leqq b^* \cap c = d$, contradiction.

This completes the proof of 5.7.

5.8. Claim. *Suppose B is a countable atomless* B.A. *absolutely omitting every $p \in \Gamma$, where $|\Gamma| \leqq \aleph_0$. Let I be a maximal ideal of B generated by $I_0 \cup I_1$, where I_0, I_1 are non-principal ideals and $I_0 \cap I_1 = \{0\}$.*

Then there is a countable atomless extension B^1 of B and a $t \in B^1 - B$ such that:

(1) *B^1 absolutely omits every $p \in \Gamma$.*

(2) *B^1 absolutely omits $p_1 = \{x \cap b = c : b, c \in I_0,\ b \cap t = c;\ or\ b \in I_1,\ c = 0\}$ and $p_2 = \{x \cup b = c : b, c \in I_0,\ b - t = c;\ or\ b \in I_1,\ c = 0\}$.*

Remark. If B^2 is an extension of B^1 omitting p_1 and p_2, then there is no maximal ideal I^1 of B^2 and non-principal ideals I_0^1, I_1^1 such that $I_0^1 \cup I_1^1$ generates I^1, $I_0^1 \cap I_1^1 = \{0\}$, and $I_l^1 \cap B = I_l$ for $l = 0, 1$. In fact, otherwise t or $-t$ is in I^1 and hence has the form $t_0 \cup t_1$ with $t_l \in I_l^1$ for $l = 0, 1$; but then t_0 realizes p_1 or p_2.

Proof of 5.8. Let G satisfy (1), (2), (3), Fact 5.6A in 5.6 and its proof, and suppose $(x \cap b^* = c^*) \in P_I$. Let $c \leqq b_0$ in $B[G]$, $\beta < \omega$; write c and b_0 as in Fact 5.6A. We want to find $x \cap b^{**} = c^{**} \in P$ extending $x \cap b^* = c^*$ so that $(x \cap b^{**} = c^{**}) \in G \Rightarrow B[G]$ satisfies the desired condition (2) for p_1, c, b_0; similarly for p_2. In some cases p_1 and p_2 can be treated simultaneously. Let B^0 be as in the proof of 5.7; again we may assume that $u \leqq b^*$ whenever $u \in B^0 \cap I$ and $[e \neq 0 \Rightarrow e = -b^*]$, $[e_0 \neq 0 \Rightarrow e_0 = -b^*]$. For each $u \in I$ write $u = g(u) \cup h(u)$ with $g(u) \in I_0$ and $h(u) \in I_1$. Now choose $a_0, \cdots, a_{2\beta-1}$ disjoint from b^* such that $a_{2i+l} \in I_l$, all a_i's non-zero and pairwise disjoint. This is possible since I_0 and I_1 are non-principal. We consider several cases defining $b^{**}, c^{**}, b_0^i, c^i$ $(i < \beta)$ so that if $x \cap b^{**} = c^{**} \in G$, then $c^i b_0^i$ $(i < \beta)$ exemplify 5.5's criterion in $B[G]$. For proving the existence of ϑ^i let (for a specific $i < \beta$) $B[G] \subseteq B^1$, $x \in B^1$, $x \cap b_0^i = c^i$, assume x realizes p_1 (or p_2) and we shall get a contradiction.

Case 1. $e = e_0 \in D$. Let

$$b^{**} = b^* \cup \bigcup_{i<2\beta} a_i, \qquad c^{**} = c^* \cup \bigcup_{i<\beta} a_{2i+\zeta},$$

$$b_0^i = b_0 \cup a_{2i} \cup a_{2i+1}, \qquad c^i = c \cup a_{2i+1}.$$

Suppose $x \cap b_0^i = c^i$; we claim that $x \cap h(b^{**}) \neq 0$, contradiction for both p_1 and p_2. (Note that $d_0 \leqq b^*$, so $a_{2i+1} \cap d_0 = 0$ and hence $b_0 - c \leqq b_0^i - c^i$.) In fact, $a_{2i+1} \leqq c^i \leqq x$, so $a_{2i+1} \leqq x \cap h(b^{**})$, as desired. When we refer to Case 1, we shall mean with $\zeta = 1$.

Case 2. $e_0 = 0$, d, $e \leqq d_0$, $e \in D$, $\xi = 0$. Let b^{**}, c^{**} be as in Case 1, $b_0^i = b_0$, $c^i = c$. Since $-e \leqq b^*$, we have $a_n \leqq e$ for all n. Again suppose $x \cap b_0^i = c^i$; we show $x \cap h(b^{**}) \neq 0$. We have $t \cap h(b^{**}) = h(c^{**})$ so $a_1 \leqq h(c^{**}) \leqq t$, hence $a_1 \leqq e \cap t \leqq x$; so $a_1 \leqq x$, and $a_1 \leqq b^{**} \cap a_1 \in I$, hence $a_1 \leqq h(b^{**})$, so $0 < a_1 \leqq x \cap h(b^{**})$ as desired.

Case 3. $e_0 = 0$, d, $e \leqq d_0$, $e \in D$, $\xi = 1$. Let b^{**} be as in Case 1, $c^{**} = c^* \cup \bigcup_{i<\beta} a_{2i}$, $b_0^i = b_0$, $c^i = c$.

Suppose $x \cap b_0^i = c^i$. Now $t \cap h(b^{**}) = h(c^{**}) = h(c^*)$, so $a_1 \leqq e \cap -t \leqq x$ and $x \cap h(b^{**}) \neq 0$.

Case 4. $e = e_0 = 0$ and $c \in I$, $b_0 \in B - I$. Then c, $-b_0 \leqq b^*$, so $a_n \leqq b_0 - c$ for all n. Let b^{**}, c^{**}, b_0^i, c^i be as in Case 3. If $x \cap b_0^i = c^i$ and $x \cap g(b^{**}) = g(c^{**})$, then $a_{2i} \leqq g(c^{**}) \leqq x$ and $a_{2i} \leqq b_0$, so $a_{2i} \leqq x \cap b_0^i = c^i$, contradiction; this takes care of p_1. For p_2, let b^{**}, c^{**}, b_0^i, c^i be as in Case 2. Note that $g(b^{**}) \cap -t = g(b^{**}) - g(c^{**})$. If $g(b^{**}) \cap x = g(b^{**}) - g(c^{**})$ and $x \cap b_0^i = c^i$ then $a_{2i} \leqq g(b^{**}) - g(c^{**}) \leqq x$ and $a_{2i} \leqq b_0$, so $a_{2i} \leqq x \cap b_0^i = c^i$, contradiction.

Case 5. $e = e_0 = 0$ and b_0, $c \in B - I$. Then $-c \leqq b^*$, so $a_n \leqq c$ for all n. For p_1, let b^{**}, c^{**}, b_0^i, c^i be as in Case 2. If $x \cap b_0^i = c^i$ and $x \cap g(b^{**}) = g(c^{**})$, then $a_{2i} \cap g(c^{**}) = 0$ but $a_{2i} \leqq x \cap g(b^{**})$, contradiction. For p_2, let b^{**}, c^{**}, b_0^i, c^i be as in Case 3; if $x \cap b_0^i = c^i$ and $x \cap g(b^{**}) = g(b^{**}) - g(c^{**})$ (like Case 4), then $a_{2i} \leqq c$ so $a_{2i} \leqq x \cap g(b^{**})$ and $a_{2i} \leqq g(c^{**})$, contradiction.

Case 6. $e = e_0 = 0$ and b_0, $c \in I$. Trivial.

Case 7. $e = 0$, $d \leqq d_0$, $e_0 \in D$, $\zeta = 0$. Let b^{**}, b_0^i, c^i be as in Case 1, $c^{**} = c^*$. Then for any $i < 2\beta$, $t \cap a_i = t \cap a_i \cap b^{**} = a_i \cap c^{**} = 0$. Hence $b_0 - c \leqq b_0^i - c^i$ and we can finish as in Case 1.

Case 8. $e=0$, $d \leqq d_0$, $e_0 \in D$, $\zeta=1$. Let b^{**}, b_0^i, c^i be as in Case 1, $c^{**}=c^* \cup \bigcup_{i<2\beta} a_i$. Then $a_i \leqq t$ for all $i<2\beta$, so $b_0-c \leqq b_0^i-c^i$ and we can finish as in Case 1.

5.9. CLAIM. *Let B, Γ be as in 5.8. Suppose $d^* \in B$, $c \neq d^* \neq 1$, B_1 is a subalgebra of $B \restriction -d^*$, and h is a homomorphism from B_1 onto $B_2 = B \restriction d^*$.*

Then there exist a countable atomless extension B^1 of B and $t \in B^1 - B$ with $t \leqq d^*$ such that:

(1) *B^1 absolutely omits every $p \in \Gamma$,*

(2) *B^1 absolutely omits $p=\{\psi(x): \psi$ is quantifier-free with parameters from B, and $\{h(a): a \in B_1, B \models \neg\psi[a]\}$ has at most one member$\} \cup \{x \cap a \neq 0: a \in B_1, 0<h(a) \leqq t\} \cup \{x \cap a \neq a: a \in B_1, 0<h(a) \leqq d^*-t\}$.*

REMARK. Thus there do not exist B'', B_1'', h'' with B'' extending B^1, $(B, B_1, h, d^*) < (B'', B_1'', h'', d^*)$, and B'' omitting p. Otherwise, there is an $x \in B_1''$ such that $h(x)=t$, hence x realizes p. This is clear for the second and third parts of p. For the first part, suppose ψ is as indicated, but $B'' \models \neg\psi[x]$. Thus $B'' \models \exists v(\neg\psi(v) \wedge v \in B_1'')$, so $B \models \exists v(\neg\psi(v) \wedge v \in B_1)$. Say $y \in B_1$, $B \models \neg\psi(y)$. Then $B \models \forall v(v \in B_1 \wedge \neg\psi(v) \rightarrow h(v)=h(y))$, so this holds in B''. Hence $t=h''(x)=h(y) \in B$, contradiction to $t \in B^1-B$.

PROOF OF 5.9. Let I be a maximal ideal in B such that $-d^* \in I$. We want to show that for almost all G, if $(x \cap -d^*=0) \in G$, then the conclusion of 5.9 holds for $B^1=B[G]$. To this end, let G satisfy the conditions (1)–(3), of 5.6 and 5.6A. Let $(x \cap b^*=c^*) \in P$ be arbitrary; let b_0, c be as in 5.6A. Let $\beta<\omega$. Now we want to find an extension $x \cap b^{**}=c^{**} \in P$ of it such that for any G as above, if $(x \cap b^{**}=c^{**}) \in G$ and $(x \cap -d^*=0) \in G$ then the conclusion of 5.6 holds. We may assume that $(x \cap -d^*=0) \leqq (x \cap b^*=c^*)$. Thus $-d^* \leqq b^*$ and $-d^* \cap c^*=0$.

(1) We may assume that b_0, $c \in B$ and $d^* \leqq b_0-c$.

For, note that $(x \cap d^*=0) \in P$. Hence if $c \cap d^* \neq 0$ we can take $b^{**}=b^*$, $c^{**}=c^*$, $b_0^i=b_0=b_0$, $c^i=c$. So, assume $c \cap d^*=0$. If $d^* \cap -b_0 \neq 0$, let $y_i \leqq d^* \cap -b_0$ for $i<\beta$ be pairwise disjoint. Then let $b^{**}=b^*$, $c^{**}=c^*$, $b_0^i=b_0 \cup y_i$, $c^i=c \cup y_i$. If $x \cap b_0^i=c^i$, then $y_i \leqq x \cap d^*$, as desired. Thus we may assume that $d^* \leqq b_0$. Let $b_0=d_0 \cup (e \cap t)^\xi$, as $-d^* \in I$ w.l.o.g. $e \cap -d^*=0$, i.e. $e \leqq d^*$, hence (as $d^* \leqq b_0$) $b_0=b_0 \cup d^*=b_0 \cup ((e \cap t)^\xi \cup d^*)=b_0 \cup d^* \in B$. So, $b_0 \in B$. If $c=d \cup (e \cap t)$, then $c \cap d^* \neq 0$ unless $e \cap t=0$. If $c=d \cup (e \cap -t)$, note $e \cap -t=e \cap -d^*=t \cap -d^*$ (since $c \leqq -d^*$) so $c \in B$. Thus (1) holds.

Now let $\psi(x, c, b_0)$ be the formula $x \cap b_0 = c$. Let

$$A = \{a \in B_1 : B \models \psi(a, c, b_0)\}.$$

If h has a constant value for all $a \in A$, then $\neg \psi(x, c, b_0) \in p$; we can then take $b^{**} = b^*$, $c^{**} = c^*$, $b_0^i = b_0$, $c^i = c$. So assume there are a_1, $a_2 \in A$ with $h(a_1) \neq h(a_2)$. W.l.o.g. $h(a_1) \nleqq h(a_2)$. Let $a = a_1 - a_2$; thus $h(a) \neq 0$, $a \in B_1$, and $a \cap b_0 = 0$. Now since $B \restriction d^*$ is atomless, there are pairwise disjoint non-zero $y_l \leqq h(a)$ $(l < \omega)$. W.l.o.g. one of the following conditions holds:

(i) $\forall l(y_l \leqq c^*)$,

(ii) $\forall l(y_l \leqq b^* - c^*)$,

(iii) $\forall l(y_l \cap b^* = 0)$.

Next, we can find $z_l \in B_1$, such that $h(z_l) = y_l$ and the z_l $(l < \omega)$ are pairwise disjoint. Now we split into cases according to which of (i), (ii), (iii) holds.

Case (i). Let $b^{**} = b^*$, $c^{**} = c^*$, $c^i = c$, $b_0^i = b_0 \cup z_i$, $\vartheta^i = (x \cap z_i \neq 0)$ $(i < \beta)$. Then $\vartheta^i \in p$ by the second part of p. Since $0 < h(z_i) \leqq c^* \leqq t$, and $x \cap b_0^i = c^i \vdash \neg \vartheta^i$ since $z_i \leqq b_0^i - c^i$.

Case (ii). Let $b^{**} = b^*$, $c^{**} = c^*$, $c^i = c \cup z_i$, $b_0^i = b_0 \cup z_i$, $\vartheta^i = (x \cap z_i \neq z_i)$ $(i < \beta)$. Then $\vartheta^i \in p$ by the third part of p, since $0 < h(z_i) \leqq b^* - c^*$; thus $h(z_i) \leqq d^*$ (by definition of h), and $h(z_i) \leqq -t$ since $t \cap b^* = c^*$. Further $x \cap b_0^i = c^i \vdash x \cap z_i = z_i$.

Case (iii). Let $b^{**} = b^* \cup \bigcup_{l<\beta} y_l$, $c^{**} = c^*$, $c^i = c \cup z_i$, $b_0^i = b_0 \cup z_i$, $\vartheta^i = (x \cap z_i \neq z_i)$ $(i < \beta)$. Thus $t \cap h(z_i) = 0$ since $t \cap b^{**} = c^*$, so $\vartheta^i \in p$, and we finish as in Case (ii).

So we have proved 5.9.

Now we recall some definitions. B is *endo-rigid* if for every endomorphism f of B, if $I = \{x : x = y \cup z,\ f(y) = 0,\ \forall v \leqq z(f(v) = v)\}$, then B / I is finite. B is *indecomposable* if there do not exist non-principal ideals I_0, I_1 of B with $I_0 \cap I_1 = \{0\}$ such that $I_0 \cup I_1$ generates a maximal ideal. B is *Bonnet-rigid* if whenever f and g are homomorphisms from B to another B.A. B, with f one-to-one and g onto, then $f = g$. Every endo-rigid B.A. and every Bonnet-rigid are mono-rigid, i.e., have no non-trivial one-to-one endomorphisms.

5.10. CLAIM. *Suppose B is mono-rigid but not Bonnet-rigid. Then there exist d^*, B_1, h such that $d^* \in B$, $0 \neq d^* \neq 1$, B_1 is a subalgebra of $B \restriction -d^*$, and h is a homomorphism of B_1 onto $B \restriction d^*$.*

PROOF. Let f and g be homomorphisms of B to a B.A. B^* with f one-to-one, g onto, but $f \neq g$. If g is one-one, then $g^{-1} \circ f$ is the identity, so $f = g$, contradiction. Thus g is not one-one. Say $d^* \neq 0$, $g(d^*) = 0$. Thus $0 \neq d^* \neq 1$. For any $b \in B$ we have $g(b \cap -d^*) = g(b)$, so $k = g \upharpoonright (B \upharpoonright -d^*)$ is a homomorphism from $B \upharpoonright -d^*$ onto B^*. Let $B_1 = k^{-1} f[B]$: thus B_1 is a subalgebra of $B \upharpoonright -d^*$. For any $b \in B_1$ let $h(b) = f^{-1}k(b) \cap d^*$. Then h is a homomorphism of B_1 onto $B \upharpoonright d^*$.

5.11. THEOREM. ($\diamondsuit_{\aleph_1}$). *There is an endo-rigid, Bonnet-rigid, indecomposable* B.A. *of power* $\aleph_1$.

PROOF. We construct by induction on i a B.A. B_i and a set Γ_i such that

(a) B_i is countable and atomless, and $\langle B_i : i < \aleph_1 \rangle$ is strictly increasing and continuous.

(b) Γ_i is a countable set of quantifier-free 1-types over B_i, and $\langle \Gamma_i : i < \aleph_1 \rangle$ is increasing and continuous.

(c) B_i absolutely omits each $p \in \Gamma_i$.

First, let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit_{\aleph_1}$-sequence. Write $\omega_1 = \Delta_0 \cup \Delta_1 \cup \Delta_2$, with Δ_i pairwise disjoint and of power ω_1. Let g_i map $\omega_1 \times \omega_1$ one-one onto Δ_i for all $i < 3$.

Now we let B_0 be countable atomless, $\Gamma_0 = 0$. For $\delta < \omega_1$ limit, let $B_\delta = \bigcup_{i<\delta} B_i$, $\Gamma_\delta = \bigcup_{i<\delta} \Gamma_i$. Now we define B_{i+1} and Γ_{i+1}. In all cases, including B_0, we can take the universe of B_i to be $\in \omega_1$.

Case 1. $0 \neq A_i \subseteq \Delta_0$, $B_i = i$ (i.e. B's universe is i) $g_0^{-1}[A_i] = f$ is an endomorphism of B_i, and the ideal I of 5.7 is such that B / I is infinite. We apply 5.7 to get B_{i+1} and $\Gamma_{i+1} = \Gamma_i \cup \{p\}$.

Case 2. $0 \neq A_i \subseteq \Delta_1$, $B_i = i$, and $I_0 = \{\alpha < i : g_1(\alpha, 0) \in A_i\}$ and $I_1 = \{\alpha < i : g_1(\alpha, 1) \in A_i\}$ satisfy the conditions of 5.8. We apply 5.8 to get B_{i+1}, and $\Gamma_{i+1} = \Gamma_i \cup \{p_1, p_2\}$.

Case 3. $0 \neq A_i \subseteq \Delta_2$, $B_i = i$, and for some $d^* \in B$, $h = g_2^{-1}[A_i]$ satisfies the conditions of 5.9. We apply 5.9 to get B_{i+1}, and $\Gamma_{i+1} = \Gamma_i \cup \{p\}$.

Case 4. Otherwise, we use 5.6 to get B_{i+1}, and $\Gamma_{i+1} = \Gamma_i$.

Set $B = \bigcup_{i<\omega_1} B_i$. It is routine to check the desired conditions. For illustration, we show that B is Bonnet-rigid. Suppose not, while it is known that B is endo-rigid and hence mono-rigid. Then we get d^*, B^1 and h as in 5.10. Let

$$C = \{i < \omega_1 : i = B_i, h[B^1 \cap B_i] = B_i \upharpoonright d^*, d^* \in B_i, g_2[i \times i] = i, \text{ and } (B_i, B^1 \cap B_i, h \upharpoonright B^1 \cap B_i, d^*) < (B, B^1, h, d^*)\}.$$

Then C is closed unbounded, so choose $i \in C$ such that $g_2[f] \cap i = A_i$. Then $g_2^{-1}[A_i] = h \restriction B^1 \cap B_i$ satisfies the conditions of 5.9, and $0 \neq A_i \subseteq \Delta_2$. Hence B_{i+1} and Γ_{i+1} are obtained as in 5.9. This contradicts the remark following 5.9.

5.12. THEOREM. *It is consistent with* ZFC + $2^{\aleph_0} > \aleph_1$ *that there is an endo-rigid Bonnet-rigid indecomposable* B.A. *of power* $\aleph_1$.

PROOF. Let $P = \{(B, \Gamma) : B$ countable atomless B.A. whose set of elements is some $\delta < \omega_1$, Γ a countable set of 1-types over B which B absolutely omit$\}$ order: the natural one

$Q =$ the *product* of $2^{\aleph_0}$ Sacks real with countable support.

(We start with $V = L$.) Force with $P \times Q$, the generic set of P give naturally a B.A. of power $\aleph_1$; it is as required.

The point is that for Q, player II has a winning strategy in the following game $\mathrm{Gm}(q)$ ($q \in Q$ arbitrary).

In the nth move, player I chose a Q-name $\boldsymbol{\alpha}_n$ of an ordinal and player II chooses q_{n+1}, $q_n < q_{n+1}$, and a finite set A_n, $|A_n| < n$ such that $q_{n+1} \Vdash_Q \boldsymbol{\alpha}_n \in A_n$.

In the end of the play, player II wins if $\{q_n : m < \omega\}$ has an upper bound.

§6. Additions on B.A.

6.1. DISCUSSION. $K^{\kappa}_{\mathrm{tr}}(n)$ is defined like $K^{\kappa}_{\mathrm{ptr}}$, but with n-tuples instead of pairs:

$$\psi_{\mathrm{tr}(n)}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \cdots, \langle x_n, y_n\rangle)$$

$$= \bigwedge_{l=1}^{n} y_l = y_1 \wedge P_\kappa(y_1) \wedge \bigvee_{\alpha} \left[\bigwedge_{l} y_1 \restriction \alpha = x_l \restriction \alpha \wedge y_1(\alpha) = \langle x_1(\alpha), \cdots, x_n(\alpha)\rangle \right].$$

$K^{\omega}_{\mathrm{tr}(\omega)}$ is defined like $K^{\omega}_{\mathrm{tr}(n)}$, but in level n we get n tuples

$$\psi_{\mathrm{tr}(\omega)}(\langle x_1, y_1, \cdots, \rangle) = \bigwedge_{l} y_l$$

$$= y_1 \wedge P_\omega(y_1) \wedge \bigvee_{n} \left(\bigwedge_{l=1}^{n} y_1 \restriction n = x_l \restriction n \wedge y_1(n) = \langle x_1(n), \cdots, x_n(n)\rangle \right).$$

All the theorems from §2 on ptr work for $\mathrm{tr}(n)$, $\mathrm{tr}(\omega)$.

We define $B_{\mathrm{tr}(\alpha)}(I)$ for $\alpha \leqq \omega$, $I \in K^{\omega}_{\mathrm{tr}(\alpha)}$ as the Boolean algebra generated freely by x_η ($\eta \in I$) except that if $\eta \in I$, $l(\eta) = \omega$, $\eta(l) = \langle \alpha_0, \cdots, \alpha_{k-1}\rangle$, then $x_\eta \leqq x_{(\eta \restriction l)^\wedge \langle \alpha_0 \rangle}$, $x_\eta \cap \bigcap_{m=1}^{k-1} x_{(\eta \restriction l)^\wedge \langle \alpha_m \rangle} = 0$.

Also, the theorems from §3 on ptr hold for $\mathrm{tr}(n)$, $\mathrm{tr}(\omega)$. But in addition

6.2. CLAIM. *If* $\alpha \geqq 3$, $I \in K^{\omega}_{\mathrm{tr}(\alpha)}$, *then* $B = B_{\mathrm{tr}(\alpha)}(I)$ *satisfies the strong countable chain condition.*

PROOF. Let $a_\alpha \neq 0$ ($\alpha < \omega_1$) be $\aleph_1$ pairwise disjoint elements, let $a_\alpha = \tau_\alpha(\bar{x}_{\bar{\eta}_\alpha})$, τ_α a Boolean term, $\bar{\eta}_\alpha$ a finite sequence from I, w.l.o.g. $\tau_\alpha = \tau$, and $\bar{\eta}_\alpha = \langle \eta_{\alpha_0}, \cdots, \eta_{\alpha_{k-1}} \rangle$ without repetition, and

$$a_\alpha = \bigcap_{l<k(0)} x_{\eta_{\alpha,l}} \cap \bigcap_{k(0)\leqq l<k} (1 - x_{\eta_{\alpha,l}}).$$

So there is $n(\alpha) < \omega$ such that $l(\eta_{\alpha,l}) < \omega \Rightarrow l(\eta_{\alpha,l}) < n(\alpha)$, and $l(\eta_{\alpha,l(1)}) = l(\eta_{\alpha,l(2)}) = \omega$, $l(1) \neq l(2)$ implies $\eta_{\alpha,l(1)} \restriction n(\alpha) \neq \eta_{\alpha,l(1)} \restriction n(\alpha)$.

W.l.o.g. if $m < n(\alpha)$, $l(\eta_{\alpha,i}) = \omega$, $\eta_{\alpha,i}(m) = \langle \gamma_0, \gamma_1, \cdots \rangle$ then $(\eta_{\alpha,i} \restriction m)^\wedge \langle \gamma_j \rangle \in \{\eta_{\alpha,0}, \eta_{\alpha,1}, \cdots\}$ (for we change $\bar{\eta}_\alpha$ but not $n(\alpha)$, and then uniformize τ_i again).

Now w.l.o.g. $n(\alpha) = n^*$ for every α, and $l(\eta_{\alpha,m}) = l_m$ and (by the theorem on Δ-systems) for every $m < k$, there is $\alpha_m \leqq n^*$ such that $\eta_{\alpha,m} \restriction \alpha_m$ is constant, but either $\langle \eta_{\alpha,m}(\alpha_m) : \alpha < \omega_1 \rangle$ is an indiscernible sequence of tuples (in $(\lambda, <)$) or $\alpha_m = n^*$. We can also assume that $i_1, i_2 < k$, $\alpha, \beta < \omega_1$, $l \leqq n^*$ and $\eta_{\alpha,i_1} \restriction l = \eta_{\beta,i_2} \restriction l$ implies $\eta_{\alpha,i_1} \restriction l = \eta_{\alpha,i_2} \restriction l$.

Now we define a function h from $\{x_\eta : \eta \in I\}$ to the trivial B.A. $B_0 = \{0, 1\}$. We let $h(x_{\eta_{\alpha,l}}) = 1$ for $\alpha = 0, 1$, and $l < k(0)$, we let $h(x_{\eta_{\alpha,l}}) = 0$ if $\alpha = 0, 1$ and $l \geqq k(0)$; if $m < k(0)$, $l(\eta_{\alpha,m}) = \omega$, $\alpha < 2$, $\eta_{\alpha,m}(i) = \langle \alpha_0, \cdots \rangle$ then $h(x_{(\eta_{\alpha,m} \restriction i)^\wedge \langle \alpha_0 \rangle}) = 1$. Otherwise $h(x_\eta) = 0$. It is easy to see that h can be extended to a homomorphism h' of $B_{\mathrm{tr}(\alpha)}(I)$ into B_0 (as it respects the relations which $B_{\mathrm{tr}(\alpha)}(I)$ satisfies) and $h'(a_0) = h'(a_1) = 1$. Hence $a_0 \cap a_1 \neq 0$. This holds for any α, $\beta < \omega_1$.

6.3. THEOREM. (1) *In the theorems of* §2 (*in particular* 2.2, 2.6, 2.8), *for* $3 \leqq \alpha \leqq \omega$ *we can replace* $K^{\omega}_{\mathrm{ptr}}$, $\psi^{\omega}_{\mathrm{ptr}}$, *by* $K^{\omega}_{\mathrm{tr}(\alpha)}$, $\psi^{\omega}_{\mathrm{tr}(\alpha)}$ *and get the same conclusions.*

(2) *In the theorems of* §3 *for* $3 \leqq \alpha \leqq \omega$ *we can replace* $K^{\omega}_{\mathrm{ptr}}$, $\psi^{\omega}_{\mathrm{ptr}}$ *by* $K^{\omega}_{\mathrm{tr}(\alpha)}$, $\psi^{\omega}_{\mathrm{tr}(\alpha)}$ *and get improved conclusions. Mainly, in conclusion* 3.16, *we have to assume only "the full strong* $(\lambda, \lambda, \aleph_1, \aleph_1)$-$\psi_{\mathrm{tr}(\alpha)}$*-bigness for f which are strongly finitary on* P_ω", *and in* 3.16(1) *we get the* $\aleph_1$*-chain condition and* 3.16(1), (2), (3) *holds.*

PROOF. Easy.

6.4. THEOREM. *Suppose* K^{ω}_{tr} *has the full* $(\lambda, \lambda, \aleph_0, \aleph_0)$-$\psi_{\mathrm{tr}}$*-bigness property. Then*:

(1) *There is a Boolean algebra B of cardinality* λ, *with no non-trivial endomorphism onto itself*; *moreover, it is Bonnet-rigid* (*see below*).

(2) *If* a, $b \in B$ *are disjoint, non-zero, then there is no embedding of* $B \restriction a$ *into any homomorphic image of* $B \restriction b$.

(3) *We can find such* B_i $(i < 2^\lambda)$ *(as in* (1) *and* (2)) *so that for* $i \neq j$ *and non-zero* $a \in B_i$, $b \in B_j$ *there is no embedding of* $B_i \restriction a$ *into any homomorphic image of* $B_j \restriction b$.

REMARK. We shall use B.A. built from $B_{\mathrm{trr}}(I)$, hence has no long chains. We can go in the inverse direction using B.A. built from orders, using, e.g., Or (I) is the linear order with elements $\{x_\eta, y_\eta : \eta \in I\}$ such that:

(1) $l(\eta) < \omega$ implies $x_\eta < y_\eta$, for $\alpha < \beta$, $y_{\beta^\wedge\langle\alpha\rangle} < x_{\eta^\wedge\langle\beta\rangle}$ and $x_{\eta\restriction n} < x_\eta < y_\eta < y_{\eta\restriction n}$ for $n < l(\eta)$.

(2) $l(\eta) = \omega$ implies $x_{\eta\restriction n} < x_\eta = y_\eta < y_{\eta\restriction n}$ for $n < \omega$.

In such cases we need a parallel to Lemma 6.9, which is true.

6.5. DEFINITION. A Boolean algebra B is called Bonnet-rigid if there is no Boolean algebra B' and homomorphisms $h_l : B \to B'$ $(l = 0, 1)$ such that h_0 is one-to-one and h_1 is onto B', except when $h_0 = h_1$.

6.6. OBSERVATION. (1) If B is Bonnet-rigid then it has no onto endomorphism $\neq$ id.

(2) A Boolean algebra B is Bonnet-rigid if:

(*) For no disjoint non-zero $a, b \in B$ is there an embedding of $B \restriction a$ into a homomorphic image of $B \restriction b$.

PROOF OF 6.6. (2) Suppose $h_l : B \to B'$ $(l = 1, 0)$ contradict Bonnet-rigidity. Suppose first h_1 is not one-to-one, so for some $a \in B$, $a \neq 0$, $h_1(a) = 0$.

For any $b \in B$, $h_1(b - a) = h_1(b) - h_1(a) = h_1(b)$. So B' is a homomorphic image of $B \restriction (1 - a)$ and $B \restriction a$ can be embedded into it, so we finish.

If h_1 is one-to-one, then h_1 is an isomorphism from B onto B' hence $h_1^{-1}h_0 : B \to B$ is an embedding. It is not the identity (otherwise $h_0 = h_1$) so for some $a \in B$, a, $h_1^{-1}h(a)$ are disjoint non-zero; choose $b = h_1^{-1}h_0(a)$.

6.7. DEFINITION. For a set I of sequences of ordinals closed under initial segments, we define $B_{\mathrm{trr}}(I)$ as the Boolean algebra generated freely by $\{x_\eta : \eta \in I\}$, except that

(1) $x_{\eta^\wedge\langle\alpha\rangle} \cap x_{\eta^\wedge\langle\beta\rangle} = 0$ for $\alpha \neq \beta$,

(2) $x_\eta \geqq x_\nu$ when $\eta \lessdot \nu$,

(3) if η has finitely many immediate successors, $\{\eta^\wedge\langle\alpha_l\rangle : l < k_\eta\}$ then $x_\eta = \bigcup_l x_{\eta^\wedge\langle\alpha_l\rangle}$,

(4) if $\eta \lessdot \nu$, and every ρ, $\eta \leqq \rho \lessdot \nu$ has a unique immediate successor, then $x_\eta = x_\nu$.

6.8. CLAIM. (1) *The only atoms of* $B_{\mathrm{trr}}(I)$ *are* x_η, η *has no immediate successor, or* $\eta \lessdot \nu_1, \nu_2 \Rightarrow \nu_1, \nu_2$ *comparable.*

(2) *The set* $\{x_\eta : \eta \in I\}$ *is a dense subset of* $B_{\mathrm{trr}}(I)$.

(3) *Definition* (6.7) *is compatible with Definition* 3.1(3).

6.9. LEMMA. *If* B *is a homomorphic image of* $B_0 = B_{\mathrm{trr}}(I)$ *then* B *is isomorphic to some* $B_{\mathrm{trr}}(J)$, J *representable in* $M_{\aleph_0,\aleph_0}(I)$.

PROOF. So let W be an ideal of B_0 such that B is isomorphic to B_0/W. Let

$$I_1 = \{\eta \in I : x_\eta \notin W\};$$

I_1 is an approximation to J. (Clearly I_1 is closed under initial segments by 6.7(2).) Let

$$A_0 = \{\eta \in I_1 : \eta \text{ has } < \aleph_0 \text{ immediate successor in } I_1,\ \eta^\wedge\langle \alpha_l \rangle,\ l < m \text{ and } (x_\eta - \textstyle\bigcup_l x_{\eta^\wedge\langle\alpha_l\rangle}) \in W\},$$

$$A_1 = \{\eta \in I_1 : \eta \text{ has } < \aleph_0 \text{ immediate successor in } I_1,\ \eta^\wedge\langle \alpha_l \rangle,\ l < m, \text{ and } (x_\eta - \textstyle\bigcup_l x_{\eta^\wedge\langle\alpha_l\rangle}) \notin W\},$$

$$A_3 = \{(\eta, \nu) : \eta \in A_0, \eta \lessdot \nu, l(\nu) \text{ is limit, } x_\eta - x_{\nu\restriction i} \in W \text{ when } l(\eta) \leqq i < l(\nu) \text{ and for no } \eta' \lessdot \eta \text{ does } (\eta', \nu) \text{ have those properties}\},$$

$$A_3^1 = \{(\eta, \nu) \in A_3 : \nu \in I_1,\ x_\eta - x_\nu \notin W\}.$$

Let

$$J = I_1 \cup \{\eta^\wedge\langle \alpha \rangle : \eta \in A_1,\ \alpha \text{ minimal s.t. } \eta^\wedge\langle\alpha\rangle \notin I_1\} \cup \{\eta^\wedge\langle \alpha + 1 \rangle : (\eta, \nu) \in A_3^1,\ \eta^\wedge\langle\alpha\rangle \in I_1\}.$$

Now $B_{\mathrm{trr}}(J)$ is isomorphic to B, and the lemma is clear.

PROOF OF 6.4. We construct as in 3.4, using $B_{\mathrm{trr}}(I_\alpha)$ (i.e. $x = \mathrm{trr}$ there) but making the surgeries on atoms only. If $B = \mathrm{Sur}\langle i_\alpha, a_\alpha^* : \alpha < \lambda \rangle$, we can assume w.l.o.g.

(*) if $\nu \in I_\alpha$, then for some η, $\nu \leqq \eta \in I_\alpha$, $l(\eta) = \omega$.

Looking at the construction, it is clear that $B = B_{\mathrm{trr}}(I^*)$ where

$$I^* = \{\eta_1{}^\wedge\eta_2{}^\wedge \cdots {}^\wedge\eta_n : n < \omega,\ \eta_l \in I_{\alpha_l} \text{ for some } \alpha_l < \lambda, \text{ and for } l < n,\ l(\eta_l) = \omega \text{ and } a^*_{\alpha_{l+1}} \text{ is } x_{\eta_l} \text{ from } I_{\alpha_l}\}.$$

Now, in fact, it suffices to prove:

(**) if a, b are disjoint non-zero, B' a homomorphic image of $B \restriction b$, then $B \restriction a$ cannot be embedded into B'.

Suppose (**) fails and a, b, B' exemplify this. By Claim 6.8 and (*) there is $\eta \in I^*$, $x_\eta \leqq a$, $l(\eta)$ limit, and let $a^*_\alpha = x_\eta$. Clearly B' is also a homomorphic image of $B(1 - x_\eta)$, hence it is representable in $M^*_{\aleph_0,\aleph_0}(\Sigma_{j<\lambda, j\neq\alpha} I_j)$ and $B_{\mathrm{trr}}(I_\alpha)$ is embeddable into $B \restriction a$, hence into B'. By Lemma 6.9, $B' \cong B_{\mathrm{trr}}(I^+)$ for some I^+ representable in $M_{\aleph_0,\aleph_0}(\Sigma_{j<\lambda, j\neq\alpha} I_j)$. We can conclude

(***) $B_{\mathrm{trr}}(I_\alpha)$ is representable in $M_{\aleph_0,\aleph_0}(\Sigma_{j<\lambda, j\neq\alpha} I_j)$.

But from this the contradiction is trivial.

6.10. THEOREM. *If in the hypothesis of Theorem* 6.4 *we add* "$\lambda < \chi \leqq \lambda^{\aleph_0}$" *then in the conclusion we can replace* "*of cardinality* λ" *by* "*of cardinality* χ".

PROOF. Let $\{I^\alpha : \alpha < \lambda\}$ exemplify the $(\lambda, \lambda, \aleph_0, \aleph_0)$-bigness property. Let $\{A_\eta : \eta \in {}^{\omega >}\lambda\}$ be a family of λ pairwise disjoint subsets of λ, each of power λ. For each $\eta \in {}^{\omega}\lambda$, we can repeat the proof of 6.4 getting $B^\eta = B_{\mathrm{trr}}(I^*_\eta)$, such that if $\nu \in I^*_\eta$, $l(\nu) = \omega k$, then for some $\alpha = \alpha_\nu$, $I^\alpha = \{\rho \in {}^{\omega \geqq}\lambda : \nu^\frown\rho \in I^*_\eta\}$, and $\alpha_\nu \in A_{\eta \restriction k}$. Now for any $C \subseteq {}^{\omega}\lambda$ let B_C be the direct sum of $\{B_\eta : \eta \in C\}$. If $|C| = \chi$, B_C satisfies 6.4(2) (hence by 6.6 is Bonnet-rigid hence has no onto endomorphism $\neq$ id).

6.11 THEOREM. *Suppose* $\lambda > \aleph_0$ *is regular,* $\lambda < \chi \leqq \lambda^{\aleph_0}$ *and* B *is a* B.A. *as constructed in* 3.13 *hence satisfying* 3.13 (1), (2) (*hence* 3.13 (4)). *Then there is a* B.A. B_1, $B \subseteq B_1 \subseteq B^c$, B_1 *of power* χ, *and* B_1 *satisfies* 3.13 (2) (*hence* 3.13 (4) *and* B_1 *satisfies the* $\aleph_1$-*chain condition as* $B_1 \subseteq B^c$).

PROOF. Let $\{a_n : n < \omega\}$ be a maximal set of pairwise disjoint, non-zero, elements of B, such that $I = \{x \in B$: for some n, $x \leqq \bigcup_{l<n} a_l\}$ is a maximal ideal of B (such a_n's exist by B's construction). Clearly for $x \in I$, $B \restriction x = B_1 \restriction x$. Clearly for every n there is a free subset of $B \restriction a_n$ of power λ. Hence we can find in B^c elements x_i $(i < \chi)$ such that:

(i) $x_i \cap a_n \in B$ for every i and n,

(ii) for every distinct $i(1), \cdots, i(k) < \chi$ for every large enough $n < \omega$, $\{x_{i(l)} \cap a_n : l = 1, k\}$ is free (in $B \restriction a_n$).

Let B_1 be the subalgebra of B^c generated by $B \cup \{x_i : i < \chi\}$. Now B is dense in B^c, hence in B_1, and I is dense in B, so I is a dense ideal of B_1. Let $x, y \in B_1 - \{0\}$, $x \cap y = 0$ and suppose h is an embedding of $B_1 \restriction x$ into $B_1 \restriction y$, and we shall get a contradiction. W.l.o.g. $x \in I - \{0\}$; so $B_1 \restriction x = B \restriction x$. If $y \in I$, h

is an embedding of $B \restriction x$ into $B \restriction y$, contradiction. So w.l.o.g. $x' \leqq x \wedge x \neq 0 \Rightarrow h(x) \notin I$; hence $B \restriction x$ can be embedded into $B_1 \restriction I$, which is free, an easy contradiction.

6.12. THEOREM. *If λ is strong limit of cofinality $\aleph_0$ then*:

(1) *There is rigid* B.A. *of power λ satisfying the $\aleph_1$-chain condition*; *moreover, it is mono-rigid.*

(2) *There is a Bonnet-rigid* B.A. *of power λ.*

PROOF. Let $\lambda_n < \lambda$ be regular, $\Sigma \lambda_n = \lambda$, $(\forall \chi < \lambda_{n+1})\chi^{\lambda_n} < \lambda_{n+1}$ (we can use much less, but there is no point in it).

(1) Build B_n^0 as in 3.13, $|B_n^0| = \lambda_{n+1}$.

Now as λ_n is regular, we can choose the I_α we use for 3.13 (to satisfy as models the condition):

(*) For any uncountable sequence of finite sequences and countable A, there is an uncountable subsequence which is indiscernible (in the model) over A for quantifier free formulas. We can replace "uncountable" by "of power λ, whenever $\lambda = \mathrm{cf}\,\lambda \leqq 2^{\aleph_0}$".

Hence B_n^0 satisfies (*).

So if B_n $(n < \omega)$ are as in Claim 6.14 (below), B'_n the free product of B_n^0, B_n, then the direct sum of the B'_n is as required (B'_n is O.K.; as for the I_α we can use $(\lambda_n, \lambda_n, 2^{\aleph_0}, \aleph_0)$-$\psi_{\mathrm{tr}}$-unembeddability).

(2) The proof for Bonnet-rigid is similar.

6.13. LEMMA. *Suppose $\lambda_n = \lambda_n^{\aleph_0}$, $\lambda_n < \lambda_{n+1}$, $\lambda = \Sigma_{n<\omega}\lambda_n$. Then we can find Boolean algebras B_n $(n < \omega)$ such that*:

(a) $\| B_n \| = \lambda_n^+$.

(b) *B_n satisfies the $\aleph_1$-chain condition.*

(c) *There are no $n < \omega$, b, $c \in B_n$, $b \cap c = 0$, $b \neq 0$, $c \neq 0$, and embedding of $B_n \restriction b$ into the completion of $\Sigma_{m<\omega} B'_m$, where B'_m is B_m for $m \neq n$ and $B_m \restriction c$ for $m = n$.*

(d) *Any B', $\Sigma_m B_m \subseteq B' \subseteq (\Sigma_m B_m)^c$, is mono-rigid*; *moreover, there is no embedding of $B' \restriction b$ into $B' \restriction c$ for $b \in B'$, $b \neq 0$, $c \in B'$, $b \cap c = 0$.*

PROOF. Construct B_n as in 3.4 using $B_{\mathrm{tr}(n+3)}(I)$, $I \in K_{\mathrm{tr}(n+3)}$.

6.14. CLAIM. *There are* B.A. *B_n $(n < \omega)$ such that*:

(1) *B_n has power* $\mathrm{cf}\,(2^{\aleph_0})$, *and has a dense countable subset, hence satisfies the strong $\aleph_1$-chain condition.*

(2) *If B_m^1 are* B.A. *satisfying* (*) (*in the proof of* 6.12), *then B_n cannot be*

embedded into B'', $B'' = \Sigma_{m \neq n} B_m * B^1_m$ ($*$ — free product, Σ — direct sum), nor can $B_n \restriction a$, for $a \in B_n$, $a \neq 0$. Moreover, there is no homomorphism from any subalgebra of B'' onto any $B_n \restriction a$ ($a \neq 0$, $a \in B_n$).

PROOF. We use the internal algebra of some subsets of the reals: essentially the same thing appears in Bonnet [2], so we do not give a proof.

We now define a variant of Definition 1.2.

6.15. DEFINITION. (1) We say that $I \in K$ is $\psi(\bar{x}_0, \bar{x}_i, \cdots, \bar{x}_n)$-ind$_\chi$-unembeddable in $J \in K$ provided that: if f is a function from I to $M(J)$ *then* for some sequences $\bar{a}_i$ ($i < \chi$) from I, $l(\bar{a}_i) = l(\bar{x}_0) = l(x_1) \cdots$ letting $\bar{c}_i = f(\bar{a}_i)$, $\langle c_i : i < \chi \rangle$ is q.f. indiscernible in I [i.e. if $\phi(y_1, \cdots, y_n)$ is quantifier free, $i_1 < \cdots < i_h < \chi$, $j_1 < \cdots < j_k < \chi$,

$$I \vDash [\phi(\bar{c}_{i_1}, \cdots, \bar{c}_{i_k}) \equiv \phi(\bar{c}_{j_1}, \cdots, \bar{c}_{j_k})] \quad \text{and} \quad I \vDash \psi[\bar{a}_0, \cdots, \bar{a}_n].$$

Note that letting $\bar{a}_i = \bar{\tau}(\bar{c}_i)$, also $\langle \bar{c}_i : i < \chi \rangle$ is q.f.-indiscernible in J.

If the identity of μ, κ is not clear we write (μ, κ, ψ) instead of ψ.

(2) We add the adjective "strongly" if the embedding is into some $M^*(J)$ and for $i < \chi$, and subterms τ_1, τ_2 of τ, the truth value of $\tau_1(\bar{c}_i) <^* \tau_2(\bar{c}_j)$ depends on τ_1, τ_2 and the truth values of $i < j$, $i = j$, $j < i$ only.

(3) We define "...ind$_\chi$-bigness..." similarly.

6.16. CLAIM. *In the theorems of §2 we can add* ind$_{\aleph_0}$ *everywhere.*

PROOF. Same proof.

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